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An Introduction to Elastic Anisotropy

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Basic Stress-Strain Relations

1) Assume that no body torques exist in the medium of interest so that the stress tensor, denoted by σ_{ij}, is symmetric, i.e.

$$\sigma_{ij} = \sigma_{ji}. \tag{1}$$

2) Let ϵ_{ij} be the eulerian strain tensor, defined as

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2}$$

where u_i represents components of the displacement vector. By its very definition the strain tensor is symmetrical, i.e.

$$\epsilon_{ij} = \epsilon_{ji}. \tag{3}$$

3) In the most general case a linear relation between stress and strain will take the form of either

$$\sigma_{ij} = C_{ijk\ell}\epsilon_{k\ell}, \tag{4}$$

or

$$\epsilon_{ij} = S_{ijk\ell}\sigma_{k\ell}. \tag{5}$$

The coefficients $C_{ijk\ell}$ are usually called stiffness coefficients, while the coefficients $S_{ijk\ell}$ are usually called elastic compliance coefficients.

4) Symmetry properties of $C_{ijk\ell}$ (or $S_{ijk\ell}$).

$$C_{ijk\ell} = C_{ij\ell k} = C_{jik\ell} = C_{ji\ell k} \tag{6}$$

$$C_{ijk\ell} = C_{k\ell ij} \tag{7}$$

To show (6) write

$$\sigma_{ij} = C_{ijk\ell}\epsilon_{k\ell}. \tag{8}$$

Using (3), (7) becomes

$$\sigma_{ij} = C_{ijk\ell}\epsilon_{\ell k} = C_{ij\ell k}\epsilon_{k\ell}; \tag{9}$$

which implies

$$C_{ijk\ell} = C_{ij\ell k}. \tag{10}$$

In an analogous way all of (6) may be established, and (6) may be extended to include the elastic compliance coefficients.

A strain energy argument is used to establish (7). Consider an element of unstrained material in the form of a unit cube. If stresses are now introduced to produce a small homogeneous strain $\Delta\epsilon_{ij}$ given by

$$\Delta\epsilon_{ij} = \frac{1}{2}\left[\frac{\partial(\Delta u_i)}{\partial x_j} + \frac{\partial(\Delta u_j)}{\partial x_i}\right], \tag{11}$$

in the infinitesimal limit let

$$\Delta\epsilon_{ij} \to d\epsilon_{ij}. \tag{12}$$

If the strain occurs at constant temperature, then the increase in Gibbs free energy per unit volume is given by

$$dF = \sigma_{ij} d\epsilon_{ji} = \sigma_{ij} d\epsilon_{ij}. \qquad (13)$$

Now using (8),

$$dF = C_{ijk\ell} \epsilon_{k\ell} d\epsilon_{ij}, \qquad (14)$$

or

$$\frac{\partial F}{\partial \epsilon_{ij}} = C_{ijk\ell} \epsilon_{k\ell}. \qquad (15)$$

Taking the partial derivative of (15) with respect to $\epsilon_{k\ell}$ yields

$$\frac{\partial}{\partial \epsilon_{k\ell}}\left(\frac{\partial F}{\partial \epsilon_{ij}}\right) = C_{ijk\ell}. \qquad (16)$$

But the order of differentiation is immaterial, since Gibbs free energy is a state function, or

$$\frac{\partial}{\partial \epsilon_{k\ell}}\left(\frac{\partial F}{\partial \epsilon_{ij}}\right) = C_{ijk\ell} = \frac{\partial}{\partial \epsilon_{ij}}\left(\frac{\partial F}{\partial \epsilon_{k\ell}}\right) = C_{k\ell ij}. \qquad (17)$$

5) Transformation Properties of $C_{ijk\ell}$ (or $S_{ijk\ell}$).

$C_{ijk\ell}$ is a fourth rank tensor whose components transform according to the following rule for orthogonal transformations;

$$C'_{stuv} = C_{ijk\ell} A_{si} A_{tj} A_{uk} A_{v\ell}, \qquad (18)$$

where A_{ij} is the direction cosine between the i^{th} axis of the transformed set and the j^{th} axis of the original set of axes.

Wave Propagation in an Anisotropic Medium

1) Development of the General Equation of Motion

The equation of motion for the wave propagation problem is obtained by equating the volume force to the density times the acceleration, or

$$\frac{\partial}{\partial x_i}\sigma_{ij} = \rho\frac{d^2 u_j}{dt^2}. \qquad (19)$$

For small displacements (19) becomes

$$\frac{\partial}{\partial x_i}\sigma_{ij} = \rho\frac{\partial^2 u_j}{\partial t^2}. \qquad (20)$$

Using (4) and (2), (20) becomes

$$\frac{1}{2}\frac{\partial}{\partial x_i} C_{ijk\ell}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\right) = \rho\frac{\partial^2 u_j}{\partial t^2}. \qquad (21)$$

For a homogeneous medium assume

$$\frac{\partial}{\partial x_i} C_{ijk\ell} = 0. \qquad (22)$$

Now (21) becomes, using (6) and (22),

$$\frac{1}{2} C_{jik\ell}\left(\frac{\partial^2 u_k}{\partial x_i \partial x_\ell} + \frac{\partial^2 u_\ell}{\partial x_i \partial x_k}\right) = \rho\frac{\partial^2 u_j}{\partial t^2}. \qquad (23)$$

This is the general equation that is to be solved in order to discuss wave propagation in an anisotropic medium.

2) Propagation Modes for any given Direction in an Anisotropic Medium

Let the propagation direction be specified by the vector $\vec{k}$ such that an elemental solution of (23) for forward facing waves is of the form

$$\vec{u} = \vec{u}_o e^{i(\vec{k}\cdot\vec{x}-wt)}. \qquad (24)$$

Consider wave propagation in the x_1 direction, i.e.

$$\vec{k} = k\,\vec{a}_1, \qquad (25)$$

where $\vec{a}_1$ is a unit vector along x_1. Now using (25), (24) becomes

$$\vec{u} = \vec{u}_o e^{i(k_1 x_1 - wt)}. \qquad (26)$$

Substitution of (26) into (23) yields the set of equations

$$(C_{11jk} - \rho \frac{2}{k^2}\,\delta_{jk})u_k = 0. \qquad (27)$$

A non-trivial solution exists if and only if

$$\left| C_{11jk} - \rho \frac{w^2}{k^2}\,\delta_{jk} \right| = 0. \qquad (28)$$

Solution of (28) yields three eigenvalues

$$\lambda_i = \rho(\frac{w}{k})^2_i, \qquad (29)$$

such that the velocities for the three propagation modes are

$$\vec{v}^{(i)} = \sqrt{\frac{\lambda_i}{\rho}}\, a_1. \qquad (30)$$

Substitution of λ_i into (27) will give the particle displacements associated with each mode. Any i^{th} mode will be completele specified by the set of (30) and $\vec{u}^{(i)}$.

For the solution in any general direction the above formalism applies once the stiffness tensor $C_{ijk\ell}$ is rotated such that the x_1' axis of the transformed set of axes coincides with the desired propagation direction; except for principal directions in crystals of high symmetry this is often the less tedious method. For principal direction in crystals of high symmetry it is often easier to reformulate the solution for the desired $\vec{k}$ and substitute (24) into (23) and work out the results analogous to (27) which will yield the eigenvalues for the solution.

Example 1. Find modes of propagation in LiF in [100] direction, i.e. $\vec{k} = k\,\vec{a}_1$.

For LiF at 300°K $\rho = 2.646\ \mathrm{g/cm^3}$

$$C_{1111} = C_{2222} = C_{3333} = 1.112 \times 10^{12}\ \mathrm{dynes/cm^2},$$

$$C_{1122} = C_{1133} = C_{2233} = 0.42 \times 10^{12}\ \mathrm{dynes/cm^2},$$

$$C_{2323} = C_{3131} = C_{1212} = 0.628 \times 10^{12}\ \mathrm{dynes/cm^2},$$

$$\text{all other } C_{ijk\ell} = 0.$$

Now (28) becomes

$$\begin{vmatrix} (1.112-\lambda) & 0 & 0 \\ 0 & (0.42-\lambda) & 0 \\ 0 & 0 & (0.42-\lambda) \end{vmatrix} = 0$$

or

$$\lambda_1 = 1.112$$
$$\lambda_2 = \lambda_3 = 0.42$$

now from (30)

$$C_1 = |\vec{v}^{(1)}| = \sqrt{\frac{111.2}{2.646}\frac{\mathrm{mm}}{\mu\mathrm{sec}}} = 6.5\ \mathrm{mm}/\mu\mathrm{sec}$$

$$c_2 = |\vec{v}^{(2)}| = c_3|\vec{v}^{(3)}| = \sqrt{\frac{42}{2.646}}\ \frac{mm}{\mu sec} = 4.9\ mm/\mu sec.$$

Substitution of the eigenvalues λ_i back into (27) gives

$$\vec{u}^{(1)} = u_1\vec{a}_1;\ \ u_1 \text{ arbitrary}$$
$$\vec{u}^{(2)} = u_2\vec{a}_2;\ \ u_2 \text{ arbitrary}$$
$$\vec{u}^{(3)} = u_3\vec{a}_3;\ \ u_3 \text{ arbitrary}$$

The propagation modes for LiF at 300°K in the [100] direction, i.e. $\vec{k} = k_1\vec{a}_1$, is

Mode	Velocity mm/μ sec.	Particle Displacement
longitudinal	6.5	$\vec{u} = u_1\vec{a}_1$
shear	4.9	$\vec{u} = u_2\vec{a}_2$
shear	4.9	$\vec{u} = u_3\vec{a}_3$

Matrix Notation in Elasticity

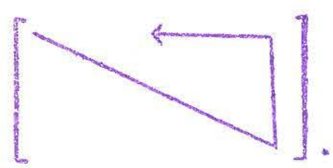
Due to the symmetries of $C_{ijk\ell}$, as given by (6) and (7), only 21 of the 81 stiffness coefficients $C_{ijk\ell}$ are independent. For the 9 components of the stress tensor, σ_{ij}, only 6 are independent; and correspondingly, for the strain tensor, ϵ_{ij}, only 6 components are independent. A matrix notation scheme is often introduced for the convenience of being able to use a "reduced" matrix multiplication scheme for the constitutive relations between stress and strain. Matrix notation is related to the tensor components as follows:

$$
\begin{aligned}
\sigma_1 &= \sigma_{11} & \epsilon_1 &= \epsilon_{11} \\
\sigma_2 &= \sigma_{22} & \epsilon_2 &= \epsilon_{22} \\
\sigma_3 &= \sigma_{33} & \epsilon_3 &= \epsilon_{33} \\
\sigma_4 &= \sigma_{23} = \sigma_{32} & \epsilon_4 &= \epsilon_{23} + \epsilon_{32} \\
\sigma_5 &= \sigma_{31} = \sigma_{13} & \epsilon_5 &= \epsilon_{31} + \epsilon_{13} \\
\sigma_6 &= \sigma_{12} = \sigma_{21} & \epsilon_6 &= \epsilon_{12} + \epsilon_{21}
\end{aligned} \qquad (31)
$$

To remember the numbering sequence for matrix notation, i.e. 11, 22, 33, 23, 31, 12, as a memonic device list the stress (or strain) tensor in matrix form as

$$
\begin{bmatrix} 11 & 12 & 13 \\ & 22 & 23 \\ & & 33 \end{bmatrix}^{*} \qquad (32)
$$

Now proceed in the direction of the arrow below,



The indices of the terms crossed will trace out the sequence

tensor notation	→	matrix notation
11		1
22		2
33		3
23		4
13		5
12		6

(33)

*The lower cross terms deleted are used to imply a symmetric matrix, i.e. only the independent elements are listed.

Now the constitutive relation between stress and strain may be written as

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}, \tag{34}$$

or

$$\sigma_i = C_{ij}\epsilon_j. \tag{35}$$

The relation between C_{mn} in matrix form and $C_{ijk\ell}$ in tensor form is such that i and j determine m, and k and ℓ determine n through (33).

Example 2.

a) Convert C_{45} to tensor notation.

By (33)

$$4 \rightarrow 23$$
$$5 \rightarrow 13$$

or

$$C_{45} = C_{2312}.$$

b) Convert C_{2113} to matrix notation.

By (6)

$$C_{1213} = C_{2113}$$

By (33)

$$12 \rightarrow 6$$
$$13 \rightarrow 5$$

or

$$C_{2113} = C_{65}$$

Through (7)

$$C_{ijk\ell} = C_{k\ell ij},$$

which implies

$$C_{mn} = C_{nm}. \qquad (36)$$

The Effect of Crystal Symmetry on $C_{ijk\ell}$

In the general case with no crystal symmetry there will be 21 independent elastic stiffness (or compliance) coefficients. Crystal symmetry operations will reduce this number. For the case of complete symmetry, i.e. the isotropic case, the number of independent coefficients is 2.

To determine the extent of crystal anisotropy look to the relation (for cubic crystals only)

$$\frac{2\,C_{44}}{C_{11} - C_{12}} = R, \qquad (37)$$

R = anisotropy ratio.
R = 1; complete elastic isotropy.
$R \neq$ 1; elastic anisotropy.

Example 3. Compare the anisotropy ratio of W and LiF.

For tungsten at 300°K.

$$C_{11} = 5.233 \times 10^{12} \text{ dynes/cm}^2$$
$$C_{12} = 2.045 \times 10^{12} \text{ dynes/cm}^2$$
$$C_{44} = 1.607 \times 10^{12} \text{ dynes/cm}^2$$

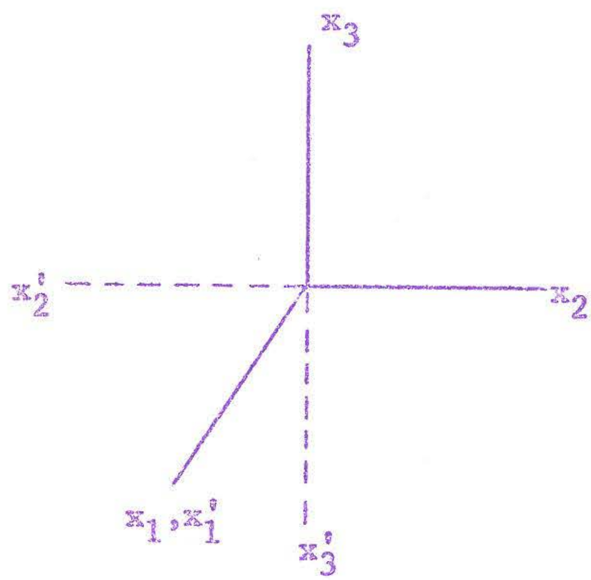
$$R_W = \frac{2(1.607)}{5.233 - 2.045} = 1.01$$

For LiF at 300°K.

$$C_{11} = 1.112 \times 10^{12} \text{ dynes/cm}^2$$
$$C_{12} = 0.42 \times 10^{12} \text{ dynes/cm}^2$$
$$C_{44} = 0.628 \times 10^{12} \text{ dynes/cm}^2$$

$$R_{LiF} = \frac{2(0.628)}{1.112 - 0.42} = 1.81$$

Example 4. Consider one 2-fold axis of symmetry about x_1; find the number of independent elastic stiffness coefficients.



A rotation as shown must not affect form of the stiffness matrix. The problem is most formally done by:

a) finding the transformation matrix,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

b) converting C_{mn} to $C_{ijk\ell}$

c) apply (18) to find the transformed elastic stiffness coefficients $C'_{ijk\ell}$.

d) convert $C'_{ijk\ell}$ to C'_{mn}

e) equate C_{mn} to C'_{mn}, set all terms to zero that contradict this equality.

The problem may also be solved by a "direct inspection" technique. Now solving this problem using the direct inspection technique.

a) the indices transform as

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow -2 \\ 3 &\rightarrow -3. \end{aligned} \tag{38}$$

b) dual sets of indices transform as products of (38), or

$$\begin{aligned} 11 &\rightarrow 11 \\ 22 &\rightarrow 22 \\ 33 &\rightarrow 33 \\ 23 &\rightarrow 23 \\ 31 &\rightarrow -31 \\ 12 &\rightarrow -12. \end{aligned} \tag{39}$$

In matrix form (39) becomes

$$\begin{aligned} 1 &\rightarrow 1 \\ 2 &\rightarrow 2 \\ 3 &\rightarrow 3 \\ 4 &\rightarrow 4 \\ 5 &\rightarrow -5 \\ 6 &\rightarrow -6 \end{aligned} \tag{40}$$

c) Now to see how C_{ij} transforms take products of (40) and substitute any changes into C'_{ij} matrix, or

$$C' = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & -C_{15} & -C_{16} \\ & C_{22} & C_{23} & C_{24} & -C_{25} & -C_{26} \\ & & C_{33} & C_{34} & -C_{35} & -C_{36} \\ & & & C_{44} & -C_{45} & -C_{46} \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{41}$$

d) Now by crystal symmetry requirements, C'_{mn} must be equal to C_{mn}, or

$$C' = C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}. \tag{42}$$

Thus a twofold axis of symmetry reduces the number of independent coefficients from 21 to 13.