PLASTIC STRAIN IN LIF LOADED
BY <111> PLANAR SHOCK

Gideon Rosenberg

Department of Physics
Washington State University
Pullman, Washington

Internal Report

77-02

Shock Dynamics Laboratory

June, 1977

The following definitions and assumptions will be used:

A. Generalized Hooke's law:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \qquad (1)$$

B. Increments of elastic and plastic strain are additive to produce the total strain:

$$d\varepsilon^{T} = d\varepsilon^{e} + d\varepsilon^{p} \qquad (2)$$

C. No volume change associated with plastic flow:

$$d\varepsilon_x^p + d\varepsilon_y^p + d\varepsilon_z^p = 0 \qquad (3)$$

D. Coordinates: Use 3 frames of reference:

(1) Crystal frame x, y, z.

(2) Shock frame x', y', z'.

(3) Glide system frame x", y", z".

1. Dislocation Glide Model

The glide systems of LiF are known:

a. The six planes [110] with the glide direction <110>. These are known as primary slip planes.

b. The three planes [100] with glide direction <110>. They are known as secondary slip systems.

It can be shown that planar shock along <111> direction does not produce shear stress acting on [110]<110> system, so in the following, we deal only with the secondary slip systems.

Assisted by Fig. 1, we write down all the possible secondary slip systems:

a. $[0,0,1]\langle\bar{1},\bar{1},0\rangle$

b. $[1,0,0]\langle 0,\bar{1},\bar{1}\rangle$

c. $[0,1,0]\langle\bar{1},0,\bar{1}\rangle$

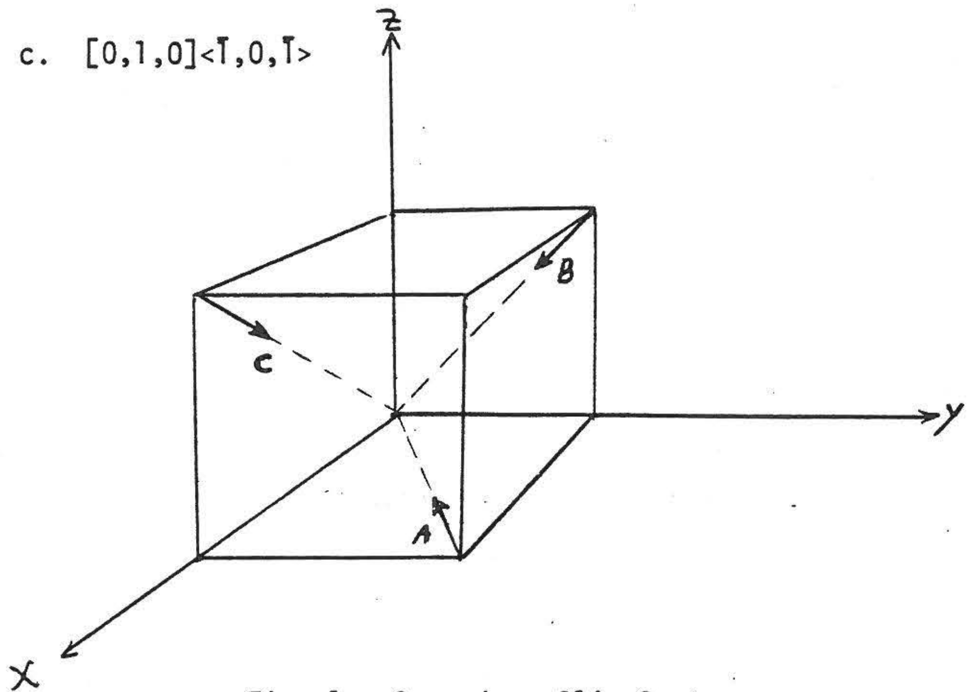


Fig. 1 - Secondary Slip Systems

Next we define for each slip system a unit vector normal to the glide plane, $\vec{n}$, and a unit vector in the slip direction, $\vec{m}$. Their cross product: $\vec{p} = \vec{m} \times \vec{n}$. We have for a., b. and c. slip systems:

a. $\vec{m} = (-1,-1,0)\frac{1}{\sqrt{2}}$

$\vec{n} = (0,0,-1)$

$\vec{p} = (1,-1,0)\frac{1}{\sqrt{2}}$

b. $\vec{m} = (0,-1,-1)\frac{1}{\sqrt{2}}$

$\vec{n} = (-1,0,0)$

$\vec{p} = (0,1,-1)\frac{1}{\sqrt{2}}$

c. $\vec{m} = (-1,0,-1)\frac{1}{\sqrt{2}}$

$\vec{n} = (0,-1,0)$

$\vec{p} = (-1,0,1)\frac{1}{\sqrt{2}}$

We also chose the x" frame in such a way that the vector $\vec{m}$ will be on the y" axis pointing in the positive direction. E.g., for system a the x" frame is one rotated at 135° counterclockwise around z axis, as illustrated in Fig. 2.

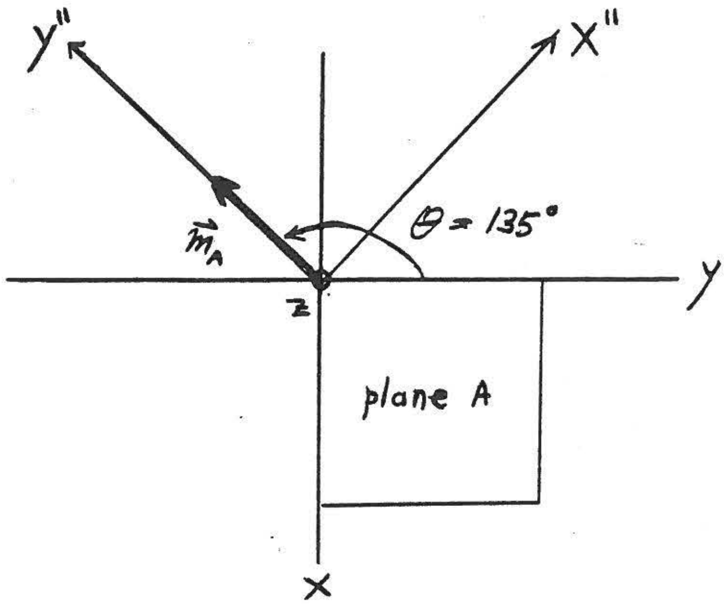


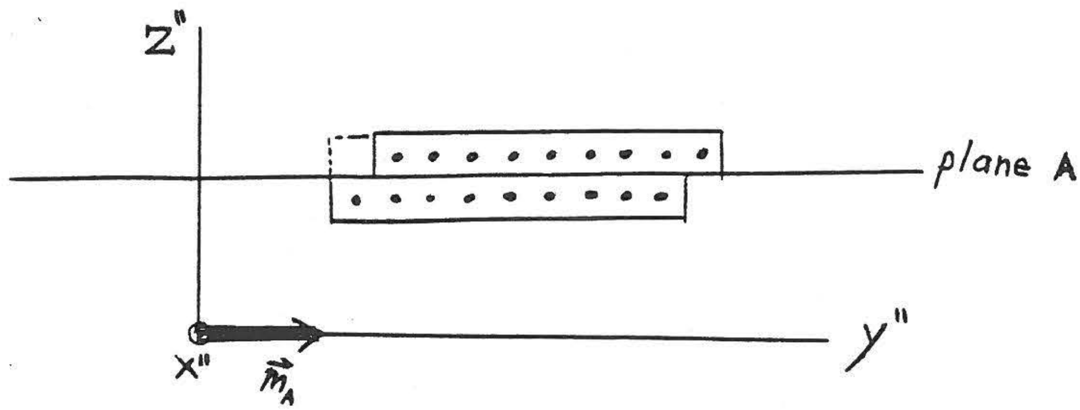
Fig. 2

This choice will make the plastic strain rate tensor in the x" frame always of the form:

$$\dot{\varepsilon}_k^{p''} = \dot{\gamma}_k \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (4)$$

where $\dot{\gamma}_k$ is the amount of strain rate, to be expressed later using a dislocation model. k is an index for the slip system, stands for A, B, or C.

To find $\dot{\gamma}_k$, we go to a microscopic picture, Fig. 3. The view is from the positive x" axis.



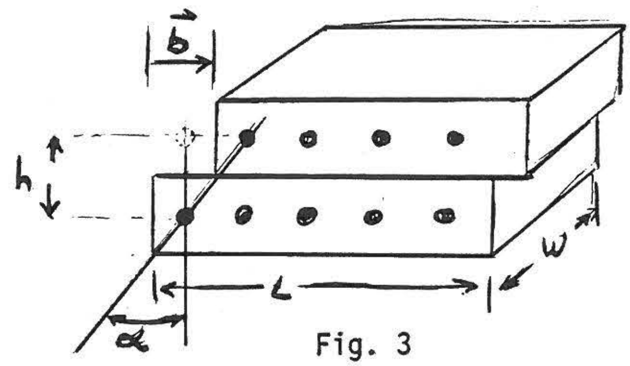


Fig. 3

Here $\vec{b}$ is the burgers vector, h is the distance between planes of atoms, α is the angular strain. Notice that:

$$\alpha = \frac{b}{h}$$

Suppose now we have a dislocation line normal to plane A which, as a result of the slip, is moving along $\vec{m}_A$ at velocity ν, moving a distance L in time t. Then:

$$\alpha = \frac{\nu bt}{Lh} = \frac{\nu btw}{Lhw}$$

Set: $w/Lhw = N_m^A$ (line per volume, is the definition for dislocation density) and the rate:

$$\dot{\alpha} = \frac{\alpha}{t}$$

we have:

$$\dot{\alpha}^k = \phi\nu bN_m^k \tag{5}$$

ϕ is a geometrical factor: $0 < \phi < 1$. Also, one can claim that $N_m^k = \frac{N_m}{3}$ where N_m is total number of etch pits per unit area counted on one {100} face. This is an argument to say that what was counted is a contribution from all 3 glide systems. Using Eq. (5) to express $\dot{\gamma}_k$ in Eq. (4), and recalling the definition of infinitesimal strain is:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

we find:

$$\dot{\gamma}^k = \frac{1}{2}\dot{\alpha}^k \tag{6}$$

2. Transformation of the Plastic Strain to the Crystal Frame.

The strain rate in the crystal x frame can be derived from the expression given in the x" frame by rotation:

$$^{k}\dot{\varepsilon}^{p}_{ij} = a_{im}a_{jn}\,{}^{k}\dot{\varepsilon}^{p''}_{mn} \tag{7}$$

substituting ${}^{k}\dot{\varepsilon}^{p''}_{mn}$ from Eq. (4) and using Eq. (6), Eq. (7) yields:

$$^{k}\dot{\varepsilon}^{p}_{ij} = \left(\frac{\dot{\alpha}^k}{2}\right)(a_{i1}a_{j2} + a_{j1}a_{i2}) \tag{8}$$

The transformation components a_{ij} are the elements of the matrix A_{ij} defined by the components of the vectors $\vec{m}$, $\vec{n}$, $\vec{p}$ as follows:

$$A_{ij} = \begin{pmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{pmatrix} \tag{9}$$

So we obtain for the 3 slip systems:

$$A^A_{ij} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{pmatrix} \tag{10A}$$

$$A^B_{ij} = \begin{pmatrix} 0 & -1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{10B}$$

$$A^C_{ij} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad (10C)$$

Doing all the algebra using Eq. (8) and Eqs. (10A, 10B, 10C) we obtain the plastic strain expressed in the crystal frame, due to slip systems A, B, C:

Plane A: $^{A}\dot{\varepsilon}^{p}_{13} = {}^{A}\dot{\varepsilon}^{p}_{23} = \frac{\dot{\alpha}^A}{2\sqrt{2}}$ and all other $\dot{\varepsilon}_{ij} = 0$.

Plane B: $^{B}\dot{\varepsilon}^{p}_{12} = {}^{B}\dot{\varepsilon}^{p}_{13} = \frac{\dot{\alpha}^B}{2\sqrt{2}}$ and all rest zeros.

Plane C: $^{C}\dot{\varepsilon}^{p}_{12} = {}^{C}\dot{\varepsilon}^{p}_{23} = \frac{\dot{\alpha}^C}{2\sqrt{2}}$ and all rest zeros.

The total plastic strain rate is:

$$\dot{\varepsilon}^p_{ij} = \sum_k {}^k\varepsilon^p_{ij} \qquad (11)$$

Using Eq. (11) to sum over A, B, C we find:

$$\dot{\varepsilon}^p_{ij} = \frac{\dot{\alpha}}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad (12)$$

Here we have also used the assumption that $\dot{\alpha}^A = \dot{\alpha}^B = \dot{\alpha}^C = \dot{\alpha}$ comes from symmetry considerations.

3. The Elastic Strain in Shock Frame, and its Transformation to Crystal Frame.

Consider a planar shock wave along <111>. Since plastic strain allowed, the requirement of uniaxial shock is one of uniaxial strain, to be applied to the total strain, means $\varepsilon^T_x \neq 0$ and all rest $\varepsilon^T_{ij} = 0$. Hence, we imply for the <u>elastic</u> strain only the requirement that there be no shear deformation, which is evident from symmetry considerations. The strain tensor in the x' frame is:

$$\varepsilon'^{e}_{ij} = \begin{bmatrix} \varepsilon'^{e}_{11} & 0 & 0 \\ 0 & \varepsilon'^{e}_{22} & 0 \\ 0 & 0 & \varepsilon'^{e}_{33} \end{bmatrix} \tag{13}$$

where: $\varepsilon'^{e}_{22} = \varepsilon'^{e}_{33}$

To be able to apply Hooke's law, we need to either transform Eq. (13) to the crystal frame, or convert the coeficients C_{ijkl} of Eq. (1) to the primed frame.

Conducting the first way, we apply the transformation:

$$\varepsilon^{e}_{ij} = Q\varepsilon'^{e}_{ij}Q^{T} \tag{14}$$

where Q is a rotation operator from the <111> (x') frame, to the <100> frame, given:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \tag{15}$$

Carrying the transformation we find:

$$\varepsilon^{e}_{ij} = \frac{1}{3} \begin{bmatrix} \varepsilon'_{11}+2\varepsilon'_{22} & \varepsilon'_{11}-\varepsilon'_{22} & \varepsilon'_{11}-\varepsilon'_{22} \\ & \varepsilon'_{11}+2\varepsilon'_{22} & \varepsilon'_{11}-\varepsilon'_{22} \\ \text{sym. elements} & & \varepsilon'_{11}+2\varepsilon'_{22} \end{bmatrix} \tag{16}$$

Note: All "e" superscript have been temporarily dropped.

Expressing Eq. (1) for a cubic crystal, also using the symmetry

$\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33}$

$\varepsilon_{12} = \varepsilon_{13} = \varepsilon_{23}$ obtained from Eq. (16), we have:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = (C_{11} + 2C_{12})\varepsilon_{11}$$
$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 2C_{44}\varepsilon_{12} \tag{17}$$

Substitute for ε_{11} and ε_{12} in Eq. (17) the primed strains from Eq. (16), we also find:

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \frac{1}{3}(C_{11} + 2C_{12})(\varepsilon'_{11} + 2\varepsilon'_{22})$$
$$\sigma_{xy} = \sigma_{xz} = \sigma_{yz} = \frac{2}{3}C_{44}(\varepsilon'_{11} - \varepsilon'_{22}) \tag{18}$$

4. Combined Elastic Plastic Stress-Strain Relations Transformed to the Shock Frame.

The relations obtained in the x frame are $\sigma = \sigma_{(\varepsilon^e)}$ [Eq. (17)] and the plastic strain rate $\dot{\varepsilon}^p(\dot{\alpha})$ [Eq. (12)]. By substituting $\dot{\varepsilon}^e = \dot{\varepsilon}^T - \dot{\varepsilon}^p$, and using Eq. (12) for values of $\dot{\varepsilon}^p$, we have:

$$\dot{\sigma}_{xx} = \dot{\sigma}_{yy} = \dot{\sigma}_{zz} = (C_{11} + 2C_{12})(\dot{\varepsilon}^T_{11} - \dot{\varepsilon}^p_{11}) \tag{19}$$

$$\dot{\varepsilon}^p_{11} = 0 \Rightarrow \dot{\varepsilon}^T_{11} = \dot{\varepsilon}^e_{11}$$

$$\dot{\sigma}_{xy} = \dot{\sigma}_{xz} = \dot{\sigma}_{yz} = 2C_{44}(\dot{\varepsilon}^T_{12} - \frac{\dot{\alpha}}{\sqrt{2}}) \tag{20}$$

Put Eq. (19) and Eq. (20) into one matrix form, and apply the inverse transformation to find $\dot{\sigma}'_{ij}$ in the shock frame:

$$\dot{\sigma}'_{ij} = Q^T\dot{\sigma}_{ij}Q \tag{21}$$

we find:

$$\dot{\sigma}'_{ij} = \begin{bmatrix} (C_{11}+2C_{12})\dot{\varepsilon}^T_{11}+4C_{44}(\dot{\varepsilon}^T_{12}-\frac{\dot{\alpha}}{\sqrt{2}}) & 0 & 0 \\ 0 & (C_{11}+2C_{12})\dot{\varepsilon}^T_{11}-2C_{44}(\dot{\varepsilon}^T_{12}-\frac{\dot{\alpha}}{\sqrt{2}}) & 0 \\ 0 & 0 & (C_{11}+2C_{12})\dot{\varepsilon}^T_{11}-2C_{44}(\dot{\varepsilon}^T_{12}-\frac{\dot{\alpha}}{\sqrt{2}}) \end{bmatrix} \tag{22}$$

Notice: $Tr(\dot{\sigma}'_{ij}) = Tr(\dot{\sigma}_{ij}) = 3(C_{11} + 2C_{12})\dot{\varepsilon}^T_{11}$

so the Hydrostatic pressure becomes:

$$\bar{p} = \frac{1}{3}Tr(\sigma'_{ij}) = (C_{11} + 2C_{12})\varepsilon^T_{11} \tag{23}$$

Since $\varepsilon_{11}^{T} = \varepsilon_{11}^{e}$ we might use Eq. (16) to set:

$$\bar{p} = \frac{1}{3}(C_{11} + 2C_{12})(\varepsilon_{11}^{'e} + 2\varepsilon_{22}^{'e}) \tag{24}$$

which expresses $\bar{p}$ in terms of the elastic strain in shock frame. Using condition (3) with $\varepsilon_{y}^{'p} = \varepsilon_{z}^{'p}$ to obtain:

$$\varepsilon_{y}^{'p} = -\frac{1}{2}\varepsilon_{x}^{'p} \tag{25}$$

and set $\varepsilon_{y}^{'T} = 0$

yielding:

$$\varepsilon_{11}^{'e} + 2\varepsilon_{22}^{'e} = \varepsilon_{x}^{'T} - \varepsilon_{x}^{'p} - 2\varepsilon_{y}^{'p} = \varepsilon_{x}^{'T} \tag{26}$$

which leads to:

$$\bar{p} = \frac{1}{3}(C_{11} + 2C_{12})\varepsilon_{x}^{'T} \tag{27}$$

To find $\dot{\varepsilon}_{12}^{T}$ we must transform Eq. (12) to the shock frame. Using:

$$\dot{\varepsilon}_{ij}^{'p} = Q^{T}\dot{\varepsilon}_{ij}^{p}Q \tag{28}$$

will make:

$$\dot{\varepsilon}_{ij}^{'p} = \frac{\dot{\alpha}}{\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{29}$$

Use Eq. (18) to see that:

$$3(\dot{\varepsilon}_{12}^{T} - \frac{\dot{\alpha}}{\sqrt{2}}) = (\dot{\varepsilon}_{11}^{'e} - \dot{\varepsilon}_{22}^{'e}) = (\dot{\varepsilon}_{11}^{'T} - \dot{\varepsilon}_{11}^{'p} - \dot{\varepsilon}_{22}^{'T} + \dot{\varepsilon}_{22}^{'p}) = (\dot{\varepsilon}_{x}^{'T} - \frac{2}{\sqrt{2}}\dot{\alpha} - \frac{1}{\sqrt{2}}\dot{\alpha}) =$$

$$= (\dot{\varepsilon}_{x}^{'T} - \frac{3}{\sqrt{2}}\dot{\alpha})$$

Here we have:

$$\dot{\sigma}_{x}^{'} = (C_{11} + 2C_{12})\dot{\varepsilon}_{x}^{T} + \frac{4C}{3}{}_{44}(\dot{\varepsilon}_{x}^{'T} - \frac{3}{\sqrt{2}}\dot{\alpha}) \tag{30}$$

or:

$$\dot{\sigma}_{x}^{'} = \dot{\bar{p}} + \frac{4}{3}C_{44}(\dot{\varepsilon}_{x}^{'T} - \frac{3}{\sqrt{2}}\dot{\alpha}) \tag{31}$$

5. Determination of the Shear Stress on the Glide Systems.

Looking in the matrix (16), we observe that all three slip systems deform the same amount, which should also be clear from the symmetry of the problem. Then from Eq. (18) we have the shear stress associated with each of those glide systems:

$$\tau = \sigma_{xy} = \frac{2\sqrt{2}}{3} C_{44} (\varepsilon^{e'}_{11} - \varepsilon^{e'}_{22}) \qquad (32)$$

Since we have already seen that:

$$(\dot{\varepsilon}^{'e}_{11} - \dot{\varepsilon}^{'e}_{22}) = (\dot{\varepsilon}^{'T}_{x} - \frac{3}{\sqrt{2}} \dot{\alpha}) \qquad (33)$$

we get:

$$\dot{\tau} = \frac{2\sqrt{2}}{3} C_{44} (\dot{\varepsilon}^{'T}_{x} - \frac{3}{\sqrt{2}} \dot{\alpha}) \qquad (33)$$

substitute in Eq. (31) yields:

$$\dot{\sigma}^{'}_{x} = \dot{\bar{p}} + \frac{2\dot{\tau}}{\sqrt{2}} \qquad (34)$$

or:

$$\tau = \frac{\sqrt{2}}{2} (\sigma^{'}_{x} - \bar{p}) \qquad (35)$$