EFFECTS OF LATERAL EXPANSION ON THE THERMOELECTRIC VOLTAGE GENERATED IN SHOCK

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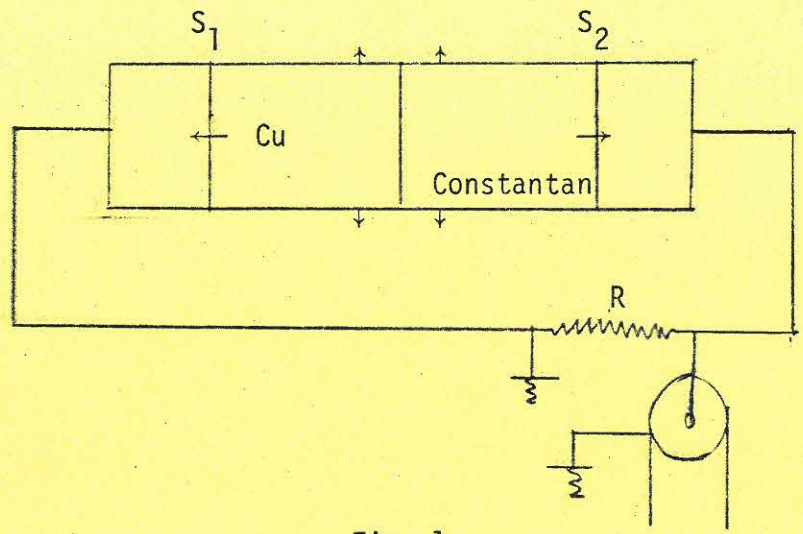

Fig. 1

The experiment is represented in somewhat idealized form in Fig. 1. Impact between Cu and constantan occurred at t = 0, producing shocks at S_1 and S_2 which are travelling as shown. Between the shocks the bars are expanding laterally at speed ν. Shocks are travelling at speed U. Material at the shock front is accelerated in the direction of motion of the shocks.

The question to be examined concerns the possibility that motion of the material between the shock fronts may cause the current flowing through the external resistance, R, to differ from that observed in the stationary case. The velocities are small compared to velocity of light, $\nu/c < 10^{-5}$, and purely relativistic effects should not be expected. In this case the effects of motion are incorporated in the two integral relations

$$-\oint \vec{E}\cdot d\vec{\ell} = \frac{d}{dt}\int_A \vec{B}\cdot d\vec{A} \tag{1}$$

$$\int \vec{J}\cdot d\vec{A} = \oint \vec{H}\cdot d\vec{\ell} \tag{2}$$

with

$$\vec{B} = \mu\vec{H}$$

$$\vec{J} = \sigma\vec{E}$$

The term on the rhs of Eq. (1) can be expanded to

$$\frac{d}{dt}\int_A \vec{B}\cdot d\vec{A} = \int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A} + \oint(\vec{v} \times \vec{B})\cdot d\vec{\ell} \qquad (3)$$

where the second integral is taken around the boundary of the region A. The second integral in (3) vanishes for $\vec{v}$ parallel to $d\vec{\ell}$. Eqs. (1) and (3) can be combined, using Stokes' theorem, to give

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{E} + \vec{v} \times \vec{B}) \qquad (4)$$

and Eq. (2) becomes

$$\nabla \times \vec{H} = \vec{J} \qquad (5)$$

Here $\vec{E}$ is electric field in the moving conductor; $\vec{E} + \vec{v} \times \vec{B}$ is the apparent field seen by the observer at rest.

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B}) \qquad (6)$$

Direct application of Eqs. (1) and (2) will provide a guide to more detailed calculations.

If the circuit of Fig. 1 includes a driving EMF, $V_0(t)$, and carries a current i(t), Eq. (1) and the circuit equations give

$$V(t) = iR + \frac{d}{dt}(iL) \qquad (7)$$

where self-inductance, L, is defined as

$$L(t) = \frac{1}{i(t)}\int \vec{B}\cdot d\vec{A} \qquad (8)$$

If L is constant, Eq. (8) is the ordinary circuit equation for series resistance and inductance. If L is made to vary by forcibly changing the geometry of the circuit, the effect is that of an additional EMF introduced into the circuit.

Consider, for example, a circuit in the form of a circular loop, as in Fig. 2. Magnetic field lines generated by the current i are directed out of the paper inside the loop and into the paper on the

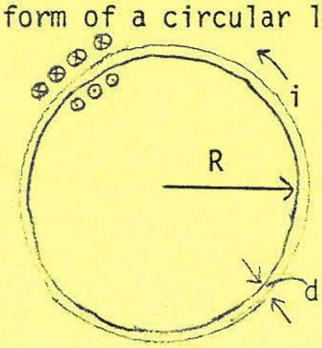

Fig. 2

outside. Take the direction of the surface normal to be out of the paper. The magnetic field at each point is proportional to current (Sommerfeld, p. 105), so

$$\int \vec{B}\cdot d\vec{A} = \mu\alpha\cdot\pi R^2\cdot\frac{2\pi i}{R} = 2\pi^2\mu i\alpha R \tag{9}$$

where $2\pi\mu i/R$ is the value of B at the center of the circle and α is a geometric factor which depends on loop radius and wire diameter, d.

If we force the loop in Fig. 2 to expand by $\nu\delta t$ with $\delta t << L/R$, the characteristic decay time for currents in the loop, the number of lines of force that are cut is $B_R\cdot 2\pi R\nu\delta t$. That is the change in the integral of Eq. (9) is

$$\delta \int \vec{B}\cdot d\vec{A} = 2\pi R\nu\delta t\cdot B_R$$

where B_R is the magnetic field at the exterior surface of the wire. For an infinitely long wire of diameter d, the magnetic field at the surface of the wire is

$$|B| = \frac{4\mu i}{d} \tag{10}$$

For d/R small, this provides a reasonable approximation to B_R. With the sign convention of Fig. 2,

$$B_R = -4\mu i/d, \tag{11}$$

so

$$\delta \int \vec{B}\cdot d\vec{A} = -8\pi\mu i R\nu\delta t/d \tag{12}$$

In effect, expansion of the loop acts to reduce the flux through the loop, and the current should increase in order to prevent this reduction. Eqs. (7), (8) and (12) yield the result that

$$\frac{dL}{dt} = -8\pi\mu R\nu/d \tag{13}$$

and that the rate of change of i is

$$\frac{1}{i}\frac{di}{dt} = \frac{[V(t)/i]-R-dL/dt}{L} \tag{14}$$

If $i = i_o = V/R$ is steady until $t = 0$, $V/i = R$, then the initial rate of increase of i is

$$\left(\frac{1}{i}\frac{di}{dt}\right)_o = -\frac{1}{L}\frac{dL}{dt} = \frac{4\nu}{\pi d} \tag{15}$$

For $\nu = 1 \mathrm{mm}/\mu\mathrm{sec}$, $d = 0.5$ mm, the r.h.s. of Eq. (14) is $8/\pi\alpha\mu\mathrm{sec}^{-1}$, corresponding to an e^{th}ing time of $\alpha\pi/8$ microseconds. According to Terman (p.52), $\alpha \simeq \ell n\, 16\, R/d - 2$. For $R = 50$ mm, $d = 0.5$ mm, $\alpha = 5.38$, so $\alpha\pi/8 \simeq 3\ \mu$seconds.

Unfortunately this calculation contradicts the result of a straightforward application of Eqs. (7)-(9). From Eqs. (8) and (9) (with $\alpha = \ell n 16R/d-2$), L is a monotonically increasing function of R:

$$L = 2\pi^2\mu R(\ell n\, \frac{16R}{d} - 2). \tag{16}$$

Consequently

$$\frac{dL}{dt} = \left(\frac{L}{R} + 2\pi^2\mu\right)\frac{dR}{dt} = \frac{L\nu}{R}\left(1 + \frac{1}{\alpha}\right) \tag{17}$$

where ν is the speed of radial expansion. Then Eq. (14) gives

$$\left(\frac{1}{i}\frac{di}{dt}\right)_{t=o} = -\frac{1}{L}\frac{dL}{dt} < 0, \tag{18}$$

indicating that current <u>decreases</u> at $t = 0$. For the constants used above ($R = 50$ mm, $\nu = 1$ mm/μsec, $\alpha = 5.38$), Eq. (17) gives

$$\frac{1}{L}\frac{dL}{dt} = \frac{\nu}{R}\left(1 + \frac{1}{\alpha}\right) = .0237/\mu\mathrm{sec} \tag{19}$$

which corresponds to a (1/e)thing time of about 40 μsec.

Eqs. (15) and (19) differ in both magnitude and sign. Consider the work done on the loop in expanding it. The energy of the magnetic field is

$$U_m = \frac{1}{2} Li^2 \tag{20}$$

If the loop under consideration has no resistance and if $V(t) \equiv 0$, but at $t = 0$, $i = i_o$, $L = L_o$, Eq. (7) reduces to

$$\frac{d}{dt}(iL) = 0$$

$$iL = \text{const.} = i_o L_o$$

$$L = \frac{i_o L_o}{i} \qquad (21)$$

Substitution of Eq. (21) into (20) gives

$$U_m(t) = \frac{1}{2} i_o L_o i(t) \qquad (22)$$

and if $i(t)$ is decreasing, so is $U_m(t)$, implying that the field is doing work on the loop in the process of expanding. This is reasonable in the following sense. The pressure exerted on a conductor by a magnetic field is proportional to B_{11}^2, where B_{11} is the field component tangent to the metal surface at the surface. Around a straight wire carrying a current, B is symmetric, so the wire is squeezed uniformly. The result of forming a loop is a slight compression of the field inside the loop because of the curvature. The result is a net force tending to expand the loop. In the process of expansion, the field does work on the loop; the magnetic energy of the field is decreased and the decrease goes into kinetic energy of the wire loop.

The result of a detailed calculation of the force on the loop will be the same as that stated in the above paragraph. In general

$$d\vec{F} = i d\vec{s} \times \vec{B} \qquad (23)$$

For the case of a circular conductor (Fig. 3) with current pointing into the paper,

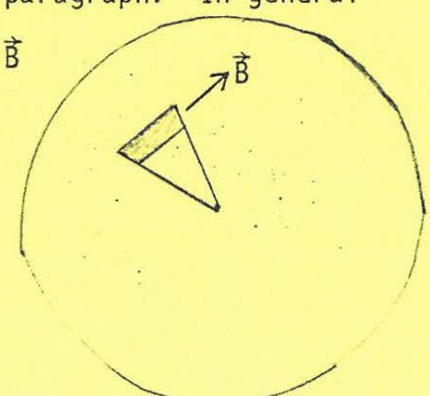

Fig. 3

$$d\vec{F} = \int JdAd\vec{s} \times \vec{B} \qquad (24)$$

$$= \int JBdsdA$$

With the force on each volume element (dAds) pointing radially inward. For a straight wire both J and B are symmetric about the axis of the wire. For the curved wire, J and B are concentrated at the inner radius, so the net force is outward.

We conclude then that the argument leading to Eq. (15) is false. The difficulty appears to arise from the assumption that the wire can cut its own lines of force. The assumption leading to Eqs. (1) and (2) is that the velocity of light is infinite. For that reason the displacement of field lines by the moving loop is immediate and the change of Eq. (12) does not occur. Instead, the magnetic flux per unit of current increases monotonically as the loop expands, and i(t) must decrease in order to maintain a balance between magnetic and kinetic energies.

Having established the validity of Eqs. (7), (16) and (17), consider a current loop in which R remains constant and d expands. Then from Eq. (16)

$$\frac{1}{L}\frac{dL}{dt} = -\frac{1}{\alpha d}\frac{dd}{dt} = -\frac{\nu}{\alpha d} \qquad (25)$$

With ν = 1 mm/μsec, α = 5.38 and d = .5 mm, as before, $\nu/\alpha d$ = 0.37 and current now does increase with time with an e-thing time of about 3 microseconds. Eq. (25) comes closer to describing the behavior sketched in Fig. 1 than does Eq. (19). It is not likely that the inductance of a real experimental circuit can be calculated with any accuracy, but it can be measured. And, with some care, it can be approximated by a sequence of static measurements in which successive estimated shapes of the changing components are used in building up a set of simulating circuit elements.

The following formulae may be of some use in estimating effects.

1. Inductance of a long straight wire of length ℓ and radius b:

$$\frac{4\pi L}{\mu} = 2\ell\left(\ln\frac{2\ell}{b} - \frac{3}{4} + \frac{4b}{\ell}\right),$$ ℓ and b in meters, L in henries

2. Self-inductance per unit length of a pair of parallel wires of radius b, separated by distance a

$$\frac{L\pi}{\mu} = \ln\frac{a}{b} + \frac{1}{4}$$

3. Same as (2), but the wires are a different radii a and b, and separation is d:

$$\frac{L\pi}{\mu} = \frac{1}{2}\ln\frac{d^2}{ab} + \frac{1}{4}$$

4. Inductance per unit length of coaxial cable:

$$L = (0.140 \lg_{10}\frac{r_2}{r_1} + .015) \times 10^{-6} \text{ per foot.}$$

r_2 and r_1 are inside radius of the outer conducter and inner radius of the outer conductor, respectively.

Other formulae and graphs will be found in Terman, Radio Engineers Handbook, pp. 47 ff.