INTERNAL REPORT SDL-85-03

DETERMINATION OF A NONLINEAR ELASTIC RELATION FOR UNIAXIAL STRAIN LOADING

Y.M. GUPTA

Fall 1985

Shock Dynamics Laboratory
Department of Physics
Washington State University
Pullman, WA 99164

In many shock wave problems, we have a need to know the elastic Hugoniot (no distinction will be made between the Hugoniot and the isentrope). I describe a method for calculating the nonlinear elastic stress-strain relation using the formulation presented by Thurston.[1,2] The method, applicable to finite strains, assumes pure mode propagation.

We consider two configurations: the final or current configuration denoted by x_1 and the initial configuration denoted by a_1. We have the following relations

$$x_i = x_i \; (a_1 \;\; a_2, \;\; a_3, \;\; t) \tag{1}$$

$$\alpha_i = x_i - a_i \tag{2}$$

$$u_i = \left(\frac{\partial x}{\partial t} \right)_{\underline{a}} \tag{3}$$

I am using α_i and u_i for displacement and particle velocity, respectively. Thurston uses u_i and v_i for these quantities.

A finite strain measure, termed Green strain, is defined as

$$\eta_{ij} = \frac{1}{2} \left(\frac{\partial x_m}{\partial a_i} \frac{\partial x_m}{\partial a_j} - \delta_{ij} \right) \tag{4}$$

For a general one-dimensional problem, variations with respect to only a_1 are of interest. For the uniaxial strain there are further restrictions; i.e., only $\alpha_1 \neq 0$. Hence, only η_{11} is non-zero.

$$\eta_{11} = \frac{\partial \alpha_1}{\partial a_1} + \frac{1}{2} \left(\frac{\partial \alpha_1}{\partial a_1} \right)^2 \tag{5}$$

An engineering strain measure, e, frequently used in shock wave studies is defined as

$$e = \frac{u}{D} = 1 - \frac{\rho_o}{\rho} \tag{6}$$

where u is particle velocity and D is the shock velocity. The second relation involving densities is written using the jump condition.

The Jacobian for the transformation is written as

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial a_1} & \frac{\partial x_2}{\partial a_1} & \frac{\partial x_3}{\partial a_1} \\ \frac{\partial x_1}{\partial a_2} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_3}{\partial a_2} \\ \frac{\partial x_1}{\partial a_3} & \frac{\partial x_2}{\partial a_2} & \frac{\partial x_3}{\partial a_3} \end{vmatrix} = \frac{\rho_o}{\rho} \tag{7}$$

For uniaxial strain, the determinant takes a very simple form and has the value $\frac{\partial x_1}{\partial a_1}$. We can then write using Eq. (6) and (2)

$$-\frac{\partial \alpha_1}{\partial a_1} = e \tag{8}$$

Substituting Eq. (8) in Eq. (5) gives

$$\eta_{11} = -e + \frac{1}{2}e^2 \tag{9}$$

The negative sign in Eq. (9) simply reflects the fact that η_{11} is positive in tension (usual convention in mechanics) while we have chosen to define e, in Eq. (6), positive in compression.

To get the stress-strain relation, we consider a series expansion of the internal energy function at constant entropy

$$\rho_o U(\eta_{ij}, S) = \rho_o U(0, S) + \frac{1}{2} C^S{}_{ijkl}\, \eta_{ij}\, \eta_{kl} + \frac{1}{6} C^S{}_{ijklmn}\, \eta_{ij}\, \eta_{kl}\, \eta_{mn} \tag{10}$$

where U is internal energy per unit mass and S is entropy per unit mass. Note, this equation is a generalization of the complete equation of state $E(S, V)$ used for fluids.

The thermodynamic stresses (called tensions by Thurston) corresponding to η_{ij}/ρ_o are:

$$t_{ij} = \rho_o \left(\frac{\partial U}{\partial \eta_{ij}} \right)_S \tag{11}$$

The isentropic elastic constants appearing in Eq. (10) are defined as

$$C^S{}_{ijkl} = \rho_o \left(\frac{\partial^2 U}{\partial \eta_{ij} \partial \eta_{kl}} \right)_S \tag{12}$$

Using Eqs. (10) and (11), we can find the relationship between t_{ij} and η_{mn}. Because of the summation implied by the repeated indices in Eq. (10), care needs to be exercised in evaluating Eq. (11). I recommend calculating t_{ij} term by term. You might want to verify the following expression:

$$t_{ij} = C^S{}_{ijkl}\, \eta_{kl} + \frac{1}{2} C^S{}_{ijklmn}\, \eta_{kl}\, \eta_{mn} \tag{13}$$

Equation (13) is the general answer we are seeking. We can make this equation more specific by considering uniaxial strain, casting it in terms of the stresses and strains used in the governing equations, and considering the crystal symmetry.

For uniaxial strain, only $\eta_{11} = 0$

$$t_{ij} = C^S{}_{ij11}\, \eta_{11} + \frac{1}{2} C^S{}_{ij11\,11}\, \eta^2{}_{11} \tag{14}$$

Using the relation between σ_{ij} and t_{ij} for the diagonal terms (Appendix), we have

$$\sigma_{11} = \frac{\rho_o}{\rho}\left[{C^S}_{11}\,\eta_{11} + \frac{1}{2}{C^S}_{11\,1}\,{\eta^2}_{11}\right] \tag{15}$$

$$\sigma_{22} = \frac{\rho}{\rho_o}\left[{C^S}_{12}\,\eta_{11} + \frac{1}{2}{C^2}_{11\,2}\,{\eta^2}_{11}\right] \tag{16}$$

$$\sigma_{33} = \frac{\rho}{\rho_o}\left[{C^S}_{13}\,\eta_{11} + \frac{1}{2}{C^S}_{11\,3}\,{\eta^2}_{11}\right] \tag{17}$$

Note, the elastic constants have been expressed in the matrix notation. η_{11} can be expressed in terms of e using Eq. (9) or the density change $\frac{\rho}{\rho_o}$ using Eq. (6).

Several comments are in order about the derivation presented here: (1) the derivation is limited to uniaxial strain, (2) the equations are valid only for isentropic processes, (3) the stress entering into the governing equations and the jump conditions is σ_{ij}, (4) extension to multi-dimensional situations requires a more in-depth understanding of the various stress measures.

Extension or incorporation of these ideas to elastic-plastic deformation is a problem that has not been uniquely resolved to date. It is a subject of considerable controversy and heated arguments among theoreticians in the field of applied mechanics. The principal difficulty appears to be the proper incorporation of rotation into the formalism. In addition, the elastic and plastic components of the strain tensor cannot be written as a sum.

Uniaxial Compression Along the C-Axis in a Hexagonal Crystal

Direction of propagation is x_3 and a_3. Hence, $x_3 = a_3 + \alpha_3$

$$\therefore\ \eta_{33} \neq 0 = \frac{\partial \alpha_3}{\partial a_3} + \frac{1}{2}\left(\frac{\partial \alpha_3}{\partial a_3}\right)^2$$

$$e = \frac{u}{D} = 1 - \frac{\rho_o}{\rho}$$

$$J = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\partial x_3}{\partial a_3} \end{vmatrix} = \frac{\rho_o}{\rho}$$

$$\therefore \frac{\partial x_3}{\partial a_3} = 1 + \frac{\partial \alpha_3}{\partial a_3} = \frac{\rho_o}{\rho}$$

or

$$1 - \frac{\rho_o}{\rho} = e = -\frac{\partial \alpha_3}{\partial a_3}$$

$$\therefore \eta_{33} = -e + \frac{1}{2} e^2$$

$$\therefore t_{ij} = C^S{}_{ij\,33}\eta_{33} + \frac{1}{2} C^2{}_{ij\,33\,33}\eta^2{}_{33}$$

Note:

$$\sigma_{11} = \frac{1}{J} \frac{\partial x_1}{\partial a_j} \frac{\partial x_1}{\partial a_i} t_{ij} = \frac{1}{J} t_{11} = \frac{\rho}{\rho_o} t_{11}$$

$$\sigma_{22} = \frac{1}{J} \frac{\partial x_2}{\partial a_j} \frac{\partial x_2}{\partial a_i} t_{ij} = \frac{1}{J} t_{22} = \frac{\rho}{\rho_o} t_{22}$$

$$\sigma_{33} = \frac{1}{J} \frac{\partial x_3}{\partial a_j} \frac{\partial x_3}{\partial a_i} t_{ij} = \frac{1}{J} \left(\frac{\partial x_3}{\partial a_3}\right)^2 t_{33} = \frac{\rho_o}{\rho} t_{33}$$

These equations could have been written directly by permuting the indices in the Appendix. Paul Horn has used these relations to derive the nonlinear compression curve for sapphire along the c-axis.

References

1. R.N. Thurston, "Wave Propagation in Fluids and Normal Solids", in *Physical Acoustics, Vol. IA,* Edited by W.P. Mason (Academic Press, 1964).

2. R.N. Thurston, "Waves in Solids", in *Handbuch Der Physik, Vol. VIa/4,* Edited by C. Truesdall (Springer-Verlag, 1974).

The first reference has an excellent presentation of the basic formalism of continuum mechanics and linear wave propagation. The first 30-50 pages should be read by everyone. The second reference is of more value to the advanced student. Reading the second reference cover to cover is a full time undertaking for a year.

APPENDIX

Relation Between $\underline{\sigma}$ and $\underline{t}$

$$\sigma_{km} = \frac{1}{J} \frac{\partial x_k}{\partial a_j} \frac{\partial x_m}{\partial a_i} t_{ij}$$

$$\sigma_{11} = \frac{1}{J} \frac{\partial x_1}{\partial a_j} \frac{\partial x_1}{\partial a_i} t_{ij} = \frac{\rho}{\rho_o} \left(\frac{\partial x_1}{\partial a_1} \right) t_{11} = \frac{\rho_o}{\rho} t_{11}$$

$$\sigma_{22} = \frac{1}{J} \frac{\partial x_2}{\partial a_j} \frac{\partial x_2}{\partial a_i} t_{ij} = \frac{\rho}{\rho_o} t_{22}$$

$$\sigma_{33} = \frac{1}{J} \frac{\partial x_3}{\partial a_j} \frac{\partial x_3}{\partial a_i} t_{ij} = \frac{\rho}{\rho_o} t_{33}$$

In deriving these equations, you want to recall

$$x_1 = a_1 + \alpha_1$$

$$x_2 = a_2$$

$$x_3 = a_3$$

and $\frac{\partial x_1}{\partial a_1} = J \frac{\rho_o}{\rho}$