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STABILITY OF NUMERICAL APPROXIMATIONS TO TIME DEPENDENT FLOWS

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INTRODUCTION

Many of the physical system of equations are difficult to solve and for a lot of these there is not analytical solution. To know the behavior of the solution of such problems, the mathematicians used the numerical methods. These are diverse (finite elements, finite differences, characteristics, fast fourier transform. . .) and discretise preferentially the space, the differential operator, the phase space. For all methods, the solution wants to be researched by a series of functions which are evaluated at each step of the calculation. These methods raise some questions which are if we call U(n) the sequences and U the solution:

1) What is the behavior of $|U(n,j) - U(jdx, dt)|$ as n tends to the infinity for dx, dt fixed?

2) What is the behavior of $|U_j^n - U(j\Delta x, \Delta t)|$ as Δx, Δt tend to zero for n, dt fixed?

3) Are they conditions for that sequences tend to the exact solution?

The answers to these questions are given by the notions of consistence and stability of the scheme which is the discrete system (u(n)) connected with he differential system of equations.

In this presentation, we talk about the necessary and sufficient condition of stability for that the sequences research the exact solution. More specifically, we will give the necessary and sifficient condition for the system of the mechanical equations with the constitutive equation of a viscous fluid [1].

This report is divided in three parts. The first of those is about the general topic of stability. We want to begin by the notion of a well posed problem because in fact hte stabilty condition for a discrete system is equivalent at the notion of well posed problem and the practical conditions for a problem be well posed are very closed than these need to the stability of a discrete sytsem. We continue by the presentation of a discrete system and extract the notions needed in the field of discrete system from an example. Then, the truncation error, the consistency and stability are exhausted. We finish this part by giving the principal theorems to find the necessary and sufficient conditions for stability. The most part of this presentation comes from the Richtmeyer theory [2] which is the basic part of this problem. But more recent works have been done, [3] [4] to perfect the notion of stability of discrete sytems.

The second part gives the framework inside which we will work, i.e., continuum mechanics in uniaxial strain, small deformations with different tbehaviors of materials and a remembering on pseudo-viscosity.

The third part presents the necessary and sufficient conditions need:

1) When the behavior is elastic plus a relaxation term and the pseudo viscosity is used (Hick's article) [5]

2) When the behavior is the Maxwell's model with constant coefficient of viscosity.

I. GENERALITY OF STABILTY CONDITIONS

I.1 Recall on well posed problem

We do a recall about this, because as we will see the conditions for the stability that will set on a discrete system are equivalent at these of a well posed problem. In addition, we will limit our talk at the linear sytem of partial equations because the stability condition is obtained by Fourier transform on the linearised system (if the system is not linear).

Let be a system of differential equations, it can be written as:

$$(1) \quad \frac{\partial u}{\partial t} = P\left(\frac{\partial}{\partial x}\right) U \begin{cases} t > 0 \in R^{+} \\ x \in R^{d} \\ U: R^{+} x\, R^{d} \rightarrow R^{m} \end{cases}$$

The operator $P\left(\frac{\partial u}{\partial x}\right)$ is defined by : $P\left(\frac{\partial u}{\partial x}\right) = \sum_{|\alpha|=r}^{r} A_\alpha D^\alpha$

where A_α are constant matrices with m x m elements belongs to R. At the system (1) is associated the inital conditions

$$(2) \quad U(x,o) = U_0(x)$$

Same for simple problem for U_0 given the system cannot have a classical solution (in the meaning of analytical solutions). So we do not know how the solution for U_0, we will take another initial condiiton $\tilde{U}_0^n$ which tends to U_0 and for which a classical solution exists. So, we can approximate the solution for the I.C.* U_0.

The second thing that we wish is: if we have two initial conditions U_{01} and U_{02} which are neighboring, so the ulterior states are also neighboring.

As we have need to define sequences U_0^n which tend to U_0 and built sequences U^n of classical solutions which must tend to U, we must work on a function space that we call ß on which one define the norm $\| \|_B$. So the two conditions described above can be resumed in a definition for well posed problem.

<u>Definition:</u> The problem at initial values (1) is well posed in ß if:

(i) There is a unique solution $U(t) = E_0(t)U_0$ for any function U_0 of D dense** in ß.

(ii) It exists two constants K>0 and μ>0 such:

$$\| E_0 (t) U_0 \| < K e^{\mu t} \|U_0\| \quad U_0 \in D$$

<u>Remark</u>: In the second condition, we see that the solution can be unbounded in time. This condition says only that for U_0 and U_0' at t given :

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*IC = Initial condition

**Dense means: It exists $U_0^n \in D$ such $U_0^n \to U_0 \in \beta$ when $n \to \infty$.

$$\| U(t) - U(t) \| < K e^{\mu t} \| U_0 - U_0 \|$$

Note: We define the Fourier transform of the system and not

$$P(ik) = \sum_{|\alpha| = 0}^{r} A (ik)^{\alpha_1} (ik)^{\alpha_2} (ik)^{\alpha_3}, \ldots, (ik)^{\alpha_n}$$

P(ik) is a m x m matrix. The system (1) can be written

$$\frac{\partial \hat{u}}{\partial t} (k,t) = P(ik) \hat{u}(k,t)$$

Theorem: The problem at initial values (1) is well posed if and only if it exists two constants $K > 0$ and $\mu > 0$ such:

$$| e^{P(ik)t} | \leq k e^{\mu t} \forall k \in R^d, t > 0$$

Remark: To use the Fourier transform, we must suppose that the functions are very regular (in fact indefinitely derivable) and in our case of application, we can have for U_0 a discontinuity. So we will have to build a sequences of U_0^n in a good space which tend to U_0 for reason that we need of space D in which U_0^n are regular and the limit of U_0^n when $n \to \infty$ is in ß space.

PROPOSITION I.2.

(Petrovsky's condition): A necessary condition for the problem (1) be well posed is that it exists a constant $\mu \geq 0$ such:

$$\Lambda(P(ik)) < \mu \quad k \in R^d$$

where $\Lambda(M) = \text{Max Re}(\lambda_i)$, eigenvalues of M. λ_i
That means that we can bound each harmonic k and the problem will not degenerate and the problem is well posed. But the condition of Petrovsky is not sufficient, why?

The reason comes from algebra and topology. When we have a matrix M, to define its norm, we use:

$$|M| = \sup_{|w| \neq 0} \frac{|Mw|}{|w|}$$

which means that we test M for all vector w possible and we take the sup of these (do not forget Mw = w is a vector).

This expression yield $|Mw| < |M|\,|w| \;\; \forall\, w$. So, if M has eignevalues λj, we can write:

$$|\lambda j w^1| < |M|\,|w^1| \Rightarrow |\lambda j| \leq |M| \;\forall\, j$$
$$\Rightarrow \zeta(M) = \max_j |\lambda j| \leq |M|$$

We see that if $\Lambda(P(ik)) \leq \mu$, this does not imply $|M| < \mu$ and that the condition does not control all the "possibilities" of the matrix.

PROPOSITION I.3

If P(ik) is a uniformly diagonalisable, i.e., it exists one reversible matrix T such for any k $|T(k)| < C1$ and $|T^{-1}(k)| < C2$, so the Petrovsky conditions is necessary and sufficient, so that the problem will be well posed.

Proof: We can write $P = T^{-1} D T$ where D is diagonal matrix and $P^n = T^{-1} D^n T$. So $e^P = T^{-1} e^D T$ and $e^{Pt} = T^{-1} e^{Dt} T$. We want bound e^{Pt}, the norm of it is:

$$|e^{Pt}| \le |T^{-1}|\ |e^{Dt}|\ |T| \le C_1 C_2 |e^{Dt}|$$

But D is diagonal matrix so:

$$|e^{Dt}| = \delta(e^{Dt}) = e^{\Lambda(Dt)} = e^{\Lambda(P)t} \Rightarrow |e^{Pt}| \le C_1 C_2 e^{\Lambda(P)t}$$

If the Petrovsky's condition is true, we have bounded $|e^{Pt}|$ and from theorem 1, we can say that the problem is well posed cq/d.

Example 1: Here is the equation:

$$\frac{\partial u}{\partial t} = a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial^2 u}{\partial \eta^2} + a_3 \frac{\partial^3 u}{\partial \eta^3} + a_4 \frac{\partial^4 u}{\partial x^4}$$

$$P(ik) = ia_1 k - a_2 k^2 - ia_3 k^3 + a_4 k^4$$

we get: $\Lambda[p(ik)] = Re(p(ik)) = k^2\ (a_4 k^2 - a_2)$

The problem at initial values is well posed if it exists $\mu \ge 0$ such $\forall\ k \in R\ k^2(a_4 k^2 - a_2) \le \mu$

it is true if $a_1 < 0$

or $a_1 = 0$ and $a_2 \geq 0$.

Example 2: Let be the linear system at constant coefficients

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{d} A_j \frac{\partial u}{\partial xj} = 0 \quad u \in R^m$$

in this case $P(ik) = i \sum_{j=1}^{d} k_j A_j$

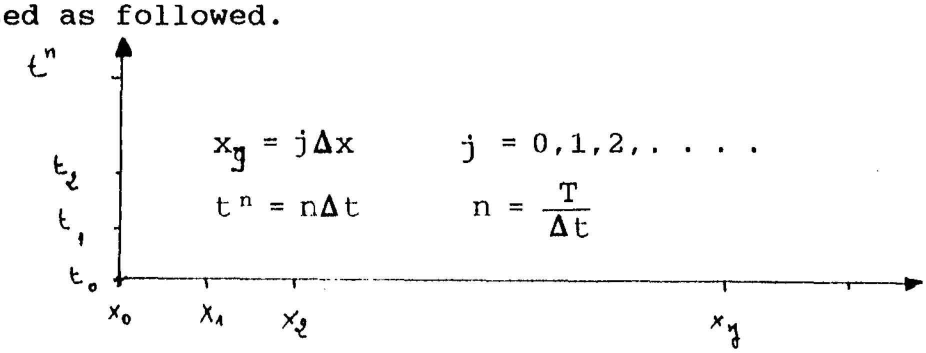
so $\Lambda[P(ik)] = \max_{e=1,m} Im[\lambda_1(k)]$

If the system is hyperbolique, i.e., the eigenvalues of A_j are real so $\Lambda\ [P(ik)] = 0$ and we can say from proposition 1 that the problem is well posed.

I.2 DISCRETE SYSTEM AND STABILITY

a) Discrete system. example.

As we said before, we limit us at the difference finite scheme which discretise the operator. To discretise an differential operator in time and space, we must in first discretise the time and the space. So, in one dimension of space, the ensemble (x,t) is discretised as followed.



The coordinnates of a point in the discrete representation are (x,t) where x = j.dx and t = n.dt. dx and dt are the incremental values of the time and space. So, the discretised function (U for example) at the point (x,t) is noted U_j^n. We must not make confusion with $U(x_j, t^n)$ which is the value of the solution function at the point (x_j, t^n).

In fact often, for physical reasons, some functions are defined at the node of cell (the velocity for example) the other at the middle of the cell (the pressure).

The discretisation of an operator is very simple and as example we can write:

$$\frac{\partial u}{\partial t} \rightarrow \frac{u_j^{n+1} - u_j^n}{dt}$$
$$\frac{\partial^2 u}{\partial x^2} \rightarrow \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2.dx}$$
$$\frac{\partial P}{\partial x} \rightarrow \frac{P_{j+1/2}^n - P_{j-1/2}^n}{dx}$$

The different types of discretisation of an operator can be found in many books, for exampel [6], [2].

Now look about a simple initial values problem, which represents the propagation of a wave with the celerity c.

$$\begin{cases} \dfrac{\partial u}{\partial t} + c\dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \quad x \in R \end{cases}$$

In Appendix A, for regulary u(x), it is shown that the problem is well posed.

A discretisation of this equaiton can be:

$$\begin{cases} \dfrac{v(x,t+\ t) - v(x,t)}{\Delta t} + C\,\dfrac{v(x+\ x,t) - v(x,t)}{\Delta x} = 0 \\ v(x,0) = u(x) \end{cases}$$

We write v in place of u to avoid confusion between the exact solution and the discrete solution.

With the discrete notation of the operator given above, the discrete system is:

$$\begin{cases} v_n^{n+1} = (1 + \eta) v_j^n - \eta v_{j+1}^n \quad \eta = c\dfrac{dt}{dx} \\ v_j^0 = u_o(x_j) \end{cases}$$

which can be written:

$$\begin{cases} v^{n+1} = s\, v^n \quad n \in N \\ v^0 = u_o \end{cases}$$

where $S = (1 + \eta)\, T_o - T$ and T_β is the operator translation $(T\ w)(x) = w(x +\ dx) \quad \beta \in Z$.

Let look some properties on this simple scheme before we give the general definitions and theorems.

*The first question which comes to the mind is, how much the discrete system approaches the real solution?

To answer at this question, define the function

$$\epsilon(a,t) = \frac{u(x,t+\Delta t) - SU(x,t)}{dt}$$

where u is the classical solution of the equation (1). Here we look S as an operator which transform u.

$$S[u(x,t)] = u(x,t) - c\frac{dt}{dx}[u(x+dx,t) - u(x,t)]$$

so $$\epsilon(x,t) = \frac{u(x,t+dt) - u(x,t)}{dt} + c\frac{u(x+dx,t) - u(x,t)}{dx}$$

If the function u is enough regulary, we can use the Taylor development and ϵ (x,t) yields:

$$\epsilon(x,t) = \frac{\partial u}{\partial t}(x,t) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^e}(x,t+\theta_1\Delta t) + c\frac{\partial u}{\partial t}(x,t) + \frac{\Delta x}{2}C\frac{\partial^2 u}{\partial x^2}(x+\theta_\ell\Delta x,t)$$

$$\forall \quad \theta_1,\theta_2 \in [0,1]$$

As u(x,t) verify $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

We see that ϵ (x,t) = 0(dt) + 0(dx)

<u>Definition I.4</u>: The function ϵ(x,t) is called the "troncation error". We will say, that the scheme of the system (2) is in first order in time and space.

<u>Remark</u>: You must very careful to reach the exact solution if you make to tend dx or dt → 0 independently. So often, you must know how dx/dt is when dx and dt tend to zero. For example, if we consider the equation $\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2}$ $\sigma > 0$ and the Dufort, Frankel's scheme [2] :

$$\frac{u_j^{nn} - u_j^{n-1}}{2\Delta t} = \sigma \frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u-j-1^n}{\Delta x^2}$$

after use the Taylor development, we get:

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\Delta t} - \sigma \frac{u_{j+1}^n - u_j^{n+1} - u_j^{n-1} + u_{j-1}^n}{(\Delta x)^2} - \left(\frac{\partial u}{\partial t} - \sigma \frac{\partial^2 u}{\partial x^2}\right)_j^2$$
$$= \left(\frac{\Delta t}{\Delta x}\right)^2 \frac{\partial^2 u}{\partial t^2} + o(\Delta t^2) + 0(\Delta x^2) + o\left(\frac{\Delta t^3}{\Delta x^2}\right)$$

We see that for the scheme "represents" the differential equation, it needs that $\frac{\partial t}{\partial x} \to 0$ as $\Delta t \to 0$. If it is not the case and if $\frac{\Delta t}{\Delta x} \to \beta$, this scheme "represents" the equation $\frac{\partial u}{\partial r} - \sigma \frac{\partial^2 x}{\partial x^2} + \sigma\beta^2 \frac{\partial^2 u}{\partial t^2} = 0$. We will say in this case that the scheme is not consistent.

*The second question is: what is the behavior of $|u_j^n - u(j\Delta x, n\Delta t)|$ as $n \to \infty$ for dx and dt fixed?

As we said in the introduction, we have now create a sequence v^n given by the recurrente equation (I.3). If we suppose that v^n is enough regular and take the fourier transform of (I.3) we obtain:

$$\left.\begin{cases} \hat{v}^{n+1} = [(1 + \eta) - \eta e^{ik\Delta x}]\, \hat{v}^n \\ \hat{v}_0 = \hat{u}_0 \end{cases}\right\} \Rightarrow \hat{v}^{n+1} = G\hat{v}^n$$

where $G = ((1 + \eta) - \eta\, e^{ik\Delta x})$ is a function of k and we see that $G > 1$ for all k. The norm of v^{n+1} in S is:

$$\|\hat{v}^{n+1}\|^2 = \int_{-\infty}^{\infty} |G(d)|^2\, |\hat{v}^{n}|^2\, dk > \int_{-\infty}^{\infty} |\hat{v}^{n}|^2 dk = \|\hat{v}^{n}\|$$

So if we use the inverse Fourier transform we find:

$$\|v_0\| < \|v_1\| < , ... , < \|v^n\|$$

This sequence is rising and is not bounded, so it cannot converge. We say that the scheme is instable.

b) <u>Generalization</u>

We spoke about the fact that dx and dt must be in relation such as if $dt \to 0$, so $dx \to 0$ too. If such is the case or we suppose that it is, $S(dx,dt) = S(dt)$. The discrete system of the (1) system of equation can be written:

$$v^{n+1} = S(dt)\,v^{n}$$
$$v^{o} = u_o$$

If we suppose the functions v^n enough regular, the fourier's transform of this system is:

$$\begin{cases} \hat{v}^{n+1}_{(k)} = G(dt,k)\,\hat{v}^{n} \\ \hat{v}^{o} = \hat{u}_o \end{cases}$$

The complex matrix G(dt,k) at mxm elements is called "symbole" of the difference operator S(dt) or amplification matrix. By successive applications, we find that:

$$\hat{v}^{n+1}(k) = G^{n}(dt,k)\hat{v}^{o}(k) \quad \text{and}$$

$$v^{n+1} = S^{n}(dt)v^{o}$$

Definition: The scheme $v^{n+1} = S(dt)v^{n}$ is called "consistency" with the system (1), if for all classical solutions $u(t)$ of (1) we have:

$$\frac{u(t) - S(dt)\,u(t)}{\Delta t} \rightarrow 0 \quad \text{when } dt \rightarrow 0$$

Definition: The scheme of the finite difference $v^{n+1} = S(dt)v^{n}$ is called "stable in S" if it exists the constants $\delta > 0$, $K_S > 0$ and $\mu_S > 0$ such:

$$\|S^{n}(dt)u_{o}\| < K_{s}e^{\mu_{s}n\,dt}\|u_{o}\| \quad \begin{cases} 0 < \Delta t < \delta\,,\ n\varepsilon N \\ u_{o}\ \varepsilon\ S \end{cases}$$

We see that the notion of stability for a scheme is equivalent at the notion of well posed problem. If v^{n} is enough regular we can use Fourier transform operation, we obtain:

Theorem: The scheme $v^{n+1} = S(dt)\,v^{n}$ is stable in S if and only if it exists $\delta > 0$, $Ks > 0$ and $\mu_S > 0$ such:

$$|G(dt,k)^{n}| < K_{s}e^{\mu_{s}n\,dt} \quad \begin{cases} k\ \varepsilon\ R^{d} \\ 0 < \Delta r < \delta \\ n\ \varepsilon\ N \end{cases}$$

So, the stability of a scheme is reduce at the study of his amplification matrix $G(dt,k)$ and again we want that this amplification must be bounded. From this theorem, one can extract directly:

Proposition: A necessary condition for the stability of a scheme is that it exists $\delta > 0$, $C > 0$ such:

$$\rho[G(\Delta t,k)] \leq 1 + C\,\Delta t \quad \begin{cases} k\ \varepsilon R^{d} \\ 0 < \Delta t < \delta \end{cases}$$

All the conditions made so far are done intrinsically for the discrete function v^n. Now, if we want to compare this function with the exact solution, we must on the constants Ks and μs make one assumption versus the constants K and μ of the real problem. That we want; is that the rise in time of the solution of the discrete problem don't exceed this of the solution of the exact problem.

We have seen a necessary condition for the stability condition of the scheme, this was not sufficient merely because the spectral radius of a matrix is smaller than the norm of this matrix. So, the different sufficient conditions that we state now (given by Ricthmeyer [2]) are linked with the general properties of the matrices.

<u>First sufficient condition</u>

If G(dt,k) is anormal matrix, the Von Neumann condition is sufficient as well as necessary for stability.

<u>Dm</u>: when a matrix is normal the norm of this matrix and its spectral radius are equal. So, if we can bound the spectral radius we can bound the norm of the matrix uniformly and by definition the scheme is stable.

cqfd

<u>Second sufficient condition</u>

If the elements of G(0,k) are bounded for all k in L and if G is lipschist-continuous at dt = 0 in the sense that

$$G(dt,k) = G(0,k) + O(dt) \quad \text{as} \quad dt \rightarrow 0,$$

where the constant implied by the expression O(dt) does not depend on k, and if $\|G(0,k)\| < 1$, the difference equations are stable.

<u>Dm</u>: The idea of the demonstration is: (see [2.])

G*G is a normal matrix and $\|G\|^2 = \text{Max } v^*G^*Gv$, so if the vectors v are normalized and μ_i are the eigenvalues of G*G the stability condition is:

$$|\mu_i| < 1 + O(dt)$$

The eigenvalues μ_i of G^*G may be powers series in fractional powers of dt. So it is preferable to have conditions directly on G(dt,k). As we know nothing about G(dt,k) when dt varies, we are looking G(dt,k) in function of G(0,k). And the better way to obtain G(dt,k) is that we can develop G in function of G(0,k). This explains the necessary notion of lipschist-continuous. The condition $\|G(0,k)\| < 1$ comes naturally to bound G(dt,k) in the same way of the von Neumann condition.

Third sufficient condition:

If it exists a constant a such that $|\Delta| > a > 0$ where Δ is the determinant of the normalized eigenvectors of G, the von Neumann condition is sufficient as well as necessary for stability.

Dm: Same if G is not normal, it can have independent eigenvectors. And, if T is the matrix of transformation of G in the eigenvectors space, we can write:

$$G = T\ P\ T^{-1} \qquad T = \text{pxp elts} \quad \text{and} \quad G^n = T\ P^n\ T^{-1}$$

Now, we want bound $\|G\|$, so for P we take the maximum of the eigenvalues and for T, we must remember that the elements of the transform matrix can be written:

$$T = \frac{\text{cofactor of T}}{\Delta}$$

As the eigenvectors can be normalized, each element of T are $|(T^{-1})_{ij}| < 1/|\Delta|$. If $R = \max|\lambda_i|$, we have:

$$|(G^n)_{ij}| < p^2/|\Delta| \cdot R^n \qquad p^2 \text{ elts in T}$$

The bound of a pxp matrix does not exceed p times the absolute value of its largest element (see [2]) and

$$\|G^n\| < \frac{p^3}{|\Delta|} \cdot R^n$$

To bound $\|G\|$, Δ must be different from zero, that imply the condition $\Delta > a > 0$.

cqfd

<u>Fourth sufficient condition</u>:

If the elements of G(dt,k) are bounded for $0 < dt < \tau$ and all k in $\mathcal{L}$ and if all the eigenvalues $\lambda(i)$ of G, with the possibility of one, lie in a circle inside the unit circle:

$$|\lambda_i| \leq \gamma < 1 \quad \begin{cases} 0 < \Delta t < \tau \\ \text{for all } k \in \mathcal{L} \\ i = 2, \ldots, p \end{cases}$$

the von Neumann condition is sufficient as well as necessary for stability.

<u>Dm</u>: The process for the demonstration is to evaluate directly the term of the matrix G^n. But for more simplicity, they use the triangular form in which appear the eigenvalues of G. So one term of the triangular matrix, when it is multiplied n times by itself, is a polynomial expression in γ if you take $\gamma = \underset{i>1}{\mathrm{Rax}} |\lambda_i|$ (λ_i eigenvalues of G). In this expression n appears only as power of γ, the other terms can be whatever. So the limitation on λ_i which must be less than + is sufficient to bound the elements of G^n.

The reason that one eigenvalue can be more than + come from those in the polynomial expression λ_1 does not appear. So there is no matter to bound it.

cqfd

CHAPTER II

II.1. Equation of Continuum Mechanics and Constitutive Equation

The system of the equations for continuum mechanics is constituted by three equations of conservation (mass, momentum, energy) and an equation of closure of the system called constitutive equation. This system of equations is a system of evolution which needs of initial and boundary conditions. In the Lagrangian frame of representation this system can be written:

$$\begin{cases} \dfrac{\partial \mu}{\partial t} + \vec{\nabla} \cdot F = 0 \\ U = \begin{pmatrix} \underline{V} \\ u \\ \xi \end{pmatrix} \qquad F = \begin{pmatrix} -\vec{u} \\ \sigma \\ u \cdot \sigma \end{pmatrix} \qquad \partial \mu = \rho_0 \, \partial x \\ \sigma = f(\vec{r}, t, T, \varepsilon, \dot{\varepsilon}, \ldots) \end{cases}$$

Where V, u, σ, E, ε, ε are respectively the specific volume, the material velocity, the stress, the energy, the strain, and the strain rate.

Often, we find the constitutive relation under the form of a function as in this system. But, in fact, this relation is incremental and its equation is a partial derivative equation. It is under this form which interests us and we will study the system associated with three types of constitutive equations.

Now, for simplicity we will note $\sigma = \sigma$; $\varepsilon = \varepsilon$.

The first is the elastic behavior:

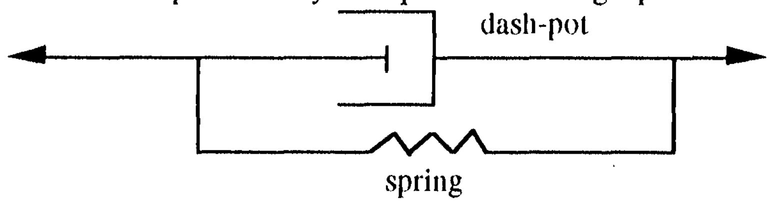
$$\frac{d\sigma}{dt} + a^2 \frac{d\varepsilon}{dv} = 0$$

The second is the visco-elastic behavior: (Kelving-Voigt's model)

$$\frac{d\sigma}{dt} + a^2 \frac{d\varepsilon}{dv} + \frac{\sigma - \sigma_e}{\tau} = 0$$

where τ = time of relaxation and σ_e = value of stress from which the stress is relaxing.

This model can be represented by the equivalent rheologic picture



The third is the viscous fluid behavior: (Maxwell's model)

$$\frac{d\sigma}{dt} + a^2 \frac{d\varepsilon}{dv} + \eta \frac{d\dot{\varepsilon}}{dt} = 0$$

Its rheologic equivalent picture is:

II.2. Limitations and Hypothesis

In this study the internal energy does not appear explicitly in the constitutive equation, so the equation of energy conservation is ignored, in plus we limit us at the uniaxial strain and isotropic materials

$$\begin{cases} \dfrac{\partial v}{\partial t} - \dfrac{\partial u}{\partial \mu} = 0 \\ \dfrac{\partial u}{\partial \varepsilon} + \dfrac{\partial \sigma}{\partial \mu} = 0 \\ \dfrac{\partial \sigma}{\partial t} + a^2 \dfrac{\partial \varepsilon}{\partial t} = 0 \end{cases}$$

or
$$\frac{\partial \sigma}{\partial \varepsilon} + a^2 \frac{\partial \varepsilon}{\partial t} + \frac{\sigma - \sigma_e}{\tau} = 0$$

or

$$\frac{\partial \sigma}{\partial t} + a^2 \frac{\partial \varepsilon}{\partial t} + \eta \frac{\partial \dot{\varepsilon}}{\partial t} = 0$$

Remark: In small deformation, the strain is given by $\varepsilon = v/v_0 - 1$ so $\frac{\partial \varepsilon}{\partial t} = \frac{1}{v_0}\frac{\partial v}{\partial t}$. From the equation of mass conservation we get $\frac{\partial v}{\partial t} = \frac{\partial u}{\partial \mu}$. If we report it in the constitutive equation the system will be reduced at two equations.

II.3. A Look on Pseudo Viscosity

We spoke in the first paragraph about the initial conditions which can be very regular (to use Fourier transform) or at least, it must exist a sequence of very regular functions which tends to this irregular function. Such irregular function exists and for our purpose that is the shock which can be represented by a step function. If we want to create a sequency of regular functions φ^n which tend to the step function, it is obvious that can be done only if $dx \to 0$. In fact, in the problem of wave propagation, we can also obtain a step function as solution same with regular initial conditions. It is the formation of a shock [7].

In a discrete problem, Δx is fixed at the beginning of the simulation. So it is impossible to catch the shock in a discrete system (now new numerical method exist to "catch" the shock). For this reason Richtmeyer and Morton are introduced a dissipative function in the system called "pseudo viscosity." The aim of this function is to spread the shock over 4 or 5 cells over the net.

The name of pseudo viscosity comes from the fact that they add at the stress tensor σ a component q which acts "as" a viscous phenomena. The general form of q is:

$$\begin{cases} q = a_1 \rho \; c \, d u + a_2 \rho |du|^2 & du < 0 \\ \quad = 0 & du > 0 \end{cases}$$

where c = sound celerity, ρ = density, du = increment of velocity, $a_1 a_2$ two constants. The first term is linear, the second is quadratic.

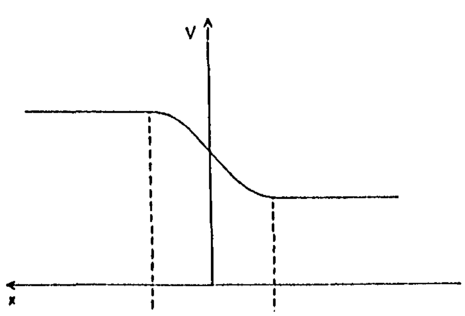


Fig. 1a

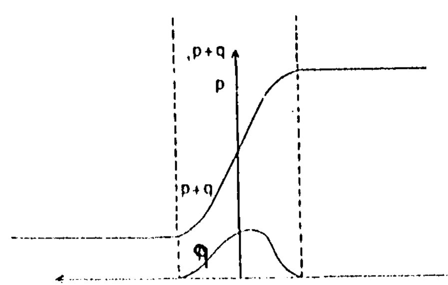


Fig. 1b

The figure 1a shows the shock spread over the net and figure 1b the body of the pseudo viscosity and its action on the stress.

One important consequence is that the system of equations will not be linear and if we want to apply the theory of the first chapter we must linearize the system.

In the next chapter, we discuss the Hick's article in which the pseudo viscosity is written under the form $q = - \wedge .du$ where $\wedge$ can be a very complex function of du.

Chapter III

STABILITY CONDITIONS FOR A SCHEME WITH PSEUDO-VISCOSITY AND A SCHEME WITH REAL VISCOSITY

III.1. Hicks's Article (Mathematics of Computation, V32, #144)

The aim of this article is to give a necessary and sufficient conditions of stability for the discrete equations of the code WONDY when pseudo-viscosity is used and the behavior of the material is as the Kelving-Voight's.

III.1.1. Equations and Notations

Taking in account the remark of the previous chapter, the system of equations in the hypothesis of small deformations and Kelving-Voight's model, can be written:

$$\text{III.1} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -\dfrac{\partial \bar{\sigma}}{\partial \mu} \\ \dfrac{\partial \sigma}{\partial t} = -a^2 \dfrac{\partial v}{\partial t} - \dfrac{\sigma - \sigma_e}{\tau} \end{cases}$$

with

$$\partial\mu = \rho_0 \partial n \; ; \quad \bar{\sigma} = \sigma + q \; ; \quad q = -\wedge du$$

The function $\wedge$ is considered as a positive constant.

<u>Discrete System</u>

In the code WONDY, the velocity is estimated at the nodes (j) and the stress in the cell (j + 1/2). So the discrete equations are:

$$\begin{cases} u_j^{n+1/2} = u_j^{n-1/2} - \left(\sigma_{j+1/2}^{n} - \sigma_{j-1/2}^{n}\right) r - \wedge r \left(u_{j+1}^{n-1/2} - 2u_j^{n-1/2} + u_{j-1}^{n-1/2}\right) \\ \sigma_j^{n+1} = \sigma_j^{n} - a^2 r \left(u_{j+1}^{n+1/2} - u_j^{n+1/2}\right) - h\left(\sigma_{j+1/2}^{n} - \sigma_e^{n}\right) \end{cases} \qquad \text{III.2}$$

$$r = \frac{\Delta t}{\Delta \mu} \qquad h = \frac{\Delta t}{\tau} \qquad a^2 = \rho_0^2 c^2$$

The stability condition of such system are find out with the substitution of the real solution by an oscillatory solution and in looking if every wavelengths can be admitted by the scheme and otherwise what are the limitations for admittance of these wavelengths.

To do that we replace the solution by a Fourier series:

$$u_j^n = \sum_k v^n(k)\, e^{ikj\Delta x} \qquad u_j^n = \begin{pmatrix} u_j^n \\ \sigma_j^n \end{pmatrix}$$

The relation must be true for each k, so for a k given, to do appear u^{n+1}, σ^{n+1} in the left side of the system, we write:

$$u_j^{n+1/2} \quad \rightarrow \quad v^{n+1}\, e^{ikj\Delta\mu}$$

$$\sigma_{j+1/2}^n - \sigma_e \quad \rightarrow \quad -a\,\omega^{n+1}\, e^{ik(j+1/2)\Delta\mu}$$

The system with this changement yields:

$$\begin{cases} \upsilon^{n+1} = (s-j)\upsilon^n + i\beta\omega^n \\ \omega^{n+1} = i\beta(i-j)\upsilon^n + (s-\beta^2-h)\omega^n \end{cases}$$

with

$$\gamma = \wedge r \left(\frac{\beta}{\alpha}\right)^2; \quad \beta = 2\alpha \sin k\frac{\Delta\mu}{2}; \quad \alpha = ar$$

Under matrix notation, we have:

$$\begin{pmatrix} \upsilon \\ \omega \end{pmatrix}^{n+1} = \begin{pmatrix} 1-\gamma & i\beta \\ i\beta(i-\gamma) & 1-\beta^2-h \end{pmatrix} \begin{pmatrix} \upsilon \\ \omega \end{pmatrix}^n$$

The amplification matrix is:

$$G(\Delta t, k) = \begin{bmatrix} 1-\gamma & i\beta \\ i\beta(1-\gamma) & 1-\beta^2-h \end{bmatrix} \qquad \text{III.3}$$

In this article, the results are given under the form of lemma for condensate reasons. For a better understanding we will make the calculations straightforward and extract as that the hypothesis which appear in this lemma .

Obviously, the matrix of amplification is not normal and our work is divided in two part. First, we use von Neumann necessary criteria; second we will use one of sufficient criteria from Richtmeyer.

Necessary Condition of Stability

Von Neumann criteria tells us that the eigenvalues of the amplification matrix must be less than 1. They are the roots of determinant of $[G - \lambda I] = 0$ which can be written:

$$\det[G(\Delta r,k) - \lambda I] = \lambda^2 - 2\lambda\left(1 - \frac{\beta^2 + h + r}{2}\right) + (1-\gamma)(1-h) \qquad \text{or}$$

$$\det[G(\Delta r,k) - \lambda I] = \lambda^2 - 2\lambda B + C \qquad (1)$$

with setting

$$B = 1 - \frac{\beta^2 + h + r}{2} \quad \text{and} \quad C = (1-\gamma)(1-h)$$

The discriminant of the equation (1) is noted D and is equal to $B^2 - C$. So, to extract the von Neumann condition we must consider three cases:

$$D > 0 \qquad \lambda_{\pm} = B \pm D^{1/2}$$

$$D < 0 \qquad \lambda_{\pm} = B \pm i\,|D|^{1/2}$$

$$D = 0 \qquad \lambda = B$$

Notation:

$$B = 1 - b \qquad b = (\beta^2 + h + r)/2$$

$$C = 1 - c \qquad c = \gamma + h - \gamma h$$

a) $D \geq 0$

The condition $|\lambda| < 1$ is equivalent at:

$$-1 < B \pm D^{1/2} < 1$$

from that we extract for B the condition:

$$\left.\begin{array}{lll} B + D^{1/2} < 1 & \Rightarrow & B < 1 \\ B - D^{1/2} > -1 & \Rightarrow & B > -1 \end{array}\right\} \Rightarrow B^2 < 1 \quad (3)$$

The condition on the first root yields:

$$-1 < B - D^{1/2} < 1$$

$$\begin{array}{ccc} D^{1/2} < B + 1 & \qquad & D^{1/2} > B - 1 \\ \Uparrow (1) & & \Uparrow \\ 2B > -(c+1) \quad (b_2) & & \text{Always true.} \end{array}$$

(b_1) and (b_2) can be written $\boxed{2|B| < C+1}$ (4)

b) $\underline{D<0}$

We have $B^2 < C$, so c is positive and $\boxed{c < 1}$ (5). The norm of the eigenvalue is $|\lambda| < C == 1 - c$. So the condition $|\ \ | < 1$ is $c > 0$ (6).

c) $\underline{D=0}$

The condition is $|\lambda| = |B| < 1$ which is equivalent at the condition 1.

If we compare with Hick's article, we can get from these results:

(1) and (2)	$\Leftrightarrow$	lemma 1A case b
(4)	$\Leftrightarrow$	lemma 1A case c

(3), (4), (5), and (6) yields $2b \geq c > 0$ and $2b + c \leq 4$ which is equivalent at lemma 1B.

Note: The lemma 1A case a) is obvious because $\lambda_t = B + D^{1/2}$ and if $B^2 > 1$ so $\lambda_+ > 1$.

The lemma 2 has been written, because in the coefficients B and C appear the variable α which gives the stability condition in terms of Δt and $\Delta\mu$ as we will see. The expression in α is an inequation of second order of type $A\alpha^2 + 2B\alpha \leq 1$.

The roots of this equation are $\alpha_{\pm} = \dfrac{-B \pm D^{1/2}}{A}$, that we can write where A is replaced by $A = B^2 - D$.

$$\begin{cases} \alpha_{+} = \dfrac{1}{D^{1/2} + B} & \alpha_{-} = -\dfrac{1}{D^{1/2} - B} \text{ when } A > 0 \text{ and} \\ \alpha_{-} = \dfrac{1}{D^{1/2} + B} & \alpha_{+} = -\dfrac{1}{D^{1/2} - B} \text{ when } A < 0. \end{cases}$$

The condition for the inequality is true if A is positive (resp negative) $\alpha < \alpha_{+}$ (resp $\alpha < \alpha$), i.e., $\alpha \ = \alpha(D^{1/2} + B) < 1$.

The restriction for the necessary condition is given under conditions $2b \geq c > 0$ by $2b + c \geq 4$, i.e.:

$$\alpha^2 + h\left(\frac{1}{2} - \wedge r\right) + 2\wedge r \leq 1.$$

The resolution of this equation yields:

- If $h \neq 0;\ \wedge \neq 0$

$$\Delta t \leq \left(1 - \frac{h}{2}\right)\left\{-\frac{\wedge}{a} + \left[\left(\frac{\wedge}{a}\right)^2 + \frac{1}{1 - h/2}\right]^{1/2}\right\}\frac{\Delta\mu}{a} \qquad \text{III.4}$$

- If $h = 0;\ \wedge \neq 0$

$$\Delta t \leq \left\{-\frac{\wedge}{a} + \left[\left(\frac{\wedge}{a}\right)^2 + 1\right]^{1/2}\right\}\frac{\Delta\mu}{a} \qquad \text{III.5}$$

- If $h = 0;\ \wedge = 0$

$$\Delta t \leq \frac{\Delta\mu}{a} \qquad \text{III.6}$$

The conditions III.5 and III.6 are respectively the Thompson and Courant Freidrichs Lewis (CFL) conditions mentioned in Hick's article.

Note: In the Thompson's condition $\wedge$ is a function of du. So, we must be careful, because the du can change from a calculation to another and the condition on Δt must take into account this variation. See [8].

Sufficient condition of stability

Hicks use to demonstrate the sufficient condition the second and the third sufficient conditions given by Richtmeyer (see Chapter I).

To use the second theorem we must show that we can write $G(\Delta t, k) = G(O, k) + O(\Delta t)$ when $\Delta t \to 0$ and $\|G(O, k)\| \leq 1$.

In the expression of $G(\Delta t, k)$ (III.3), appears $\sin^2 k \frac{\Delta t}{2}$ and h. Knowing that $h = \frac{\Delta t}{\tau}$ and so $h = O(\Delta t)$, it is sufficient to exprime $\sin^2 k \frac{\Delta \mu}{2}$ in function of k for generate the term $O(\Delta t)$. As we know that $O(\Delta t)$ goes to zero when Δt tends to zero we must take $\sin^2 k \frac{\Delta \mu}{2} < \kappa h$. That means we can find κ such that this last inequality be true. If this is the case, we can write:

$$G(\Delta t, k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -4 \wedge r\, \kappa h & 2iar(\kappa h)^{1/2} \\ 2iar(\kappa h)^{1/2}(1 - 4 \wedge r\, \kappa h) & 1 - 4\, a^2 r^2\, \kappa h - h \end{pmatrix}$$

so $G(\Delta t, k) = I + O(h) = I + O(\Delta t)$. On the other hand, $G(O, k) = \begin{bmatrix} 1 & 0 \\ 0 & 1-h \end{bmatrix}$ and $h <$ because $h = \frac{\Delta \varepsilon}{\tau}$ otherwise you cannot appreciate the relaxation, so $\|G(O, k)\| < 1$ by the

$$\text{definition } \| G \| = \frac{\mathrm{Max}|Gv|}{|v|} \quad |v| \neq 0.$$

The condition of the second theorem are satisfied for $\sin^2 k \frac{\Delta \mu}{2} < \kappa h$ and the scheme is stable.

To extend the sufficient condition for $\sin^2 k \frac{\Delta \mu}{2} > \kappa h$ the third theorem is used.

To use it, we must show that the determinant of the normalized eigenvectors is bounded by inferior values $|\Delta| \geq a > 0$.

The eigenvectors are easy to find and are:

$$V_1 = \begin{pmatrix} i\beta \\ 1-\gamma-\lambda_+ \end{pmatrix} \quad V_2 = \begin{pmatrix} -i\beta \\ 1-\gamma-\lambda_- \end{pmatrix} \quad |V_{1,2}| = \beta^2 + |1-\gamma-\lambda_\pm|^2$$

so

$$|\Delta|^2 = \frac{\beta^2}{(|V_1||V_2|)^2}(\lambda_+ - \lambda_-)^2 = \frac{4\beta^2}{(|V_1||V_2|)^2}|D|$$

Remember that γ is a function of β^2 and that $\lambda_\pm$ can be bounded by the necessary condition of D is a function of β^2. So we work in the β^2 space.

From the condition $\sin^2 k\frac{\Delta\mu}{2} > \kappa h$ taking into account the definition of β^2 we can write:

$$1 > \frac{\beta^2}{4\alpha^2} > \kappa h \quad \Rightarrow \quad \beta^2 \,\varepsilon\, [4\alpha^2 \kappa h, 4\,\alpha^2]$$

Now, we know in what interval β^2 belongs.

To bound $|\Delta|$ by lower value we can minimize $|D|$. But $D = (1 - g(\beta^2))$, so it is sufficient to take an interval where $g(\beta^2)$ is monotone increasing and less than 1.

The function $g(\beta^2)$ is: $\beta^2 g(\beta^2) = ((\beta^2 + h + \gamma)/2)^2 - \gamma h$ and one can easily show that $g(\beta^2)$ is monotone increasing for $\kappa \geq \frac{1}{4}(\alpha^2 + \wedge r)$.

In addition we want $g(\beta^2) < 1$ in this interval, i.e.,

$$g(4\alpha^2) < 1 \quad \Rightarrow \quad 0 \leq \wedge r \leq \alpha(1-\alpha) + \alpha[(1-h)^{1/2} - 1] + \frac{h}{4}.$$

If we note $\delta = \min |D|$, we get:

$$|D|^2 \geq \delta \frac{4}{\beta^2(2-\gamma)^2}$$

As we have shown, the second and third sufficient condition, so the necessary condition is also sufficient.

c. q.f.d

III.2. Scheme with Real Viscosity

We have seen that to spread the shock over the net, Richtmeyer has introduced a quantity called pseudo-viscosity. Now, we want to use a real viscosity which is a part of the behavior of the material. This behavior will be that of a Maxwell material.

2.1. Equations and Notations

As previously, the system of equations can be reduced at two equations if we suppose that the deformations are small. So, from the system (II.2) we get:

$$\begin{cases} \frac{\partial \mu}{\partial \varepsilon} = -\frac{\partial \sigma}{\partial \mu} \\ \frac{\partial \sigma}{\partial \varepsilon} = -a^2 \frac{\partial u}{\partial \mu} - \frac{\eta}{V_0} \frac{\partial}{\partial \varepsilon} \left(\frac{\partial u}{\partial \mu} \right) \end{cases} \qquad \text{III.7}$$

with $\partial\mu = \rho_0\,\partial\kappa$; η = viscosity coefficient and V_0 = initial specific volume.

2.2. Discrete System

In the same way as in 1.2, with velocity at the nodes of the net and stress at the middle of the cells, we get:

$$\begin{cases} u_j^{n+1/2} = u_j^n - (\sigma_{j+1/2}^n - \sigma_{j-1/2}^n) r \\ \sigma_{j+1/2}^{n+1} = \sigma_{j+1/2}^n - a^2 r (u_{j+1}^{n+1/2} - u_j^{n+1/2}) - \left\{ (u_{j+1}^{n+1/2} - u_j^{n+1/2}) - (u_{d+1}^{n-1/2} - u_j^{n-1/2}) \right\} \theta \end{cases} \qquad \text{III.8}$$

where

$$r = \frac{\Delta t}{\Delta \mu}, \qquad \theta = \frac{\eta}{V_0 \Delta \mu}, \qquad a^2 = \rho_0^2 c^2$$

NOTE: Here we must use r and not α as in Hick's article because $1/\Delta\mu$ appears in the second term of the second equation.

2.3. Fourier's Transform-Amplification Matrix

Identically that previously, we find the solution under the form

$$u_j^n = \sum_k V^n(k)\, e^{ikj\Delta\mu} \qquad u_j^n = \begin{pmatrix} u_j^n \\ \sigma_j^n \end{pmatrix}$$

and we set the changement:

$$u_j^{n+1/2} \rightarrow \upsilon^{n+1} e^{ikj\Delta\mu}$$

$$\sigma_{j+1/2}^{n} \dashrightarrow \omega^{n} e^{ik(j+1/2)\Delta\mu}$$

Under this changement, the system III.2.2 yields:

$$\begin{cases} \upsilon^{n+1} = \upsilon^{n} - \omega^{n} i\gamma r \\ \omega^{n+1} = \omega^{n}\left[1 - r\gamma^2(a^2 r + \theta)\right] - i a^2 r \gamma \upsilon^{n} \end{cases}$$

with $\gamma = 2 \sin \frac{h\Delta\mu}{2}$.

Under matrix notation, we have:

$$\begin{pmatrix} \upsilon \\ \omega \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & -i\,\gamma r \\ -a^2 r\, i\, \gamma & 1 - \gamma^2 r(a^2 r + \theta) \end{pmatrix} \begin{pmatrix} \upsilon \\ \omega \end{pmatrix}^{n}$$

The amplification matrix is:

$$G(\Delta t, k) = \begin{bmatrix} 1 & -i\,\gamma r \\ -a^2 r\, i\, \gamma & 1 - \gamma^2 r\,(a^2 r + \theta) \end{bmatrix} \qquad \text{III.9}$$

2.4. Necessary Condition of Stability

It is given by the knowledge of the eigenvalues of $G(\Delta t, k)$. These are solutions of the equation:

$$\det\left[G(\Delta t, k) - \lambda I\right] = \lambda^2 - 2\lambda\left(- (\gamma^2 r(a^2 r + \theta))/2\right) + (1 - \gamma^2 r\theta)$$

or

$$\det\,[G(\Delta t, k) - \lambda I] = \lambda^2 - 2\,\lambda B + C$$

with $B = 1 - \gamma^2 r\,(a^2 r + \theta)/2$; $C = 1 - \gamma^2 r\theta$.

Using the same notation as in Hick's article we get:

$$B = 1 - b \qquad b = \gamma^2 r(a^2 r + \theta)/2$$

$$C = 1 - c \qquad c = \gamma^2 r\,\theta.$$

Remembering that the necessary condition is $2b + c \leq 4$ when $2b \geq c \geq 0$ which is obvious in this case, we get:

$$\Delta t \leq \frac{\dfrac{-\eta}{V_o} + \left[\left(\dfrac{\eta}{V_o}\right)^2 + a^2 \Delta\mu^2\right]^{1/2}}{a^2} \qquad \text{III.10}$$

If we compare this expression with this obtained with pseudo-viscosity (without relaxation term III.6), they are notably different in the sense where in the III.6 expression the term in $\wedge$ (pseudo-viscosity coefficient) appears as a multiplicative coefficient in front of $\frac{\Delta\mu}{a}$ which is the CFL condition. In another way $\wedge$ is dependent of du, so Δt is dependent of du, that means dependant of the level of the shock, not the expression III.10 . So, for η fixed, you can find du for which $\Delta t_{pseudo} < \Delta t_{viscosity}$.

2.5. Sufficient Condition of Stability

We can proceed in the same way as that for system with pseudo-viscosity in using the second and third theorem of Richtmeyer, but it is more easy to use the fourth theorem on sufficient stability condition. It says that each element of $G(\Delta t, k)$ is bounded for $0 < \Delta t < \tau$ and if the eigenvalues are in the unit circle (with one exception), the von Neuman condition is sufficient.

If we look at the elements of $G(\Delta t, k)$, we get:

$$G_{11} = 1 \text{ is bounded}$$

$$G_{21} = -i\gamma r \; ; \quad |-i\gamma r| = \gamma^2 r^2 < 4\, r^2 < \frac{4}{\Delta\mu^2}\tau^2$$

$$G_{12} = -a^2 ri\gamma \; ; \quad |-a^2 ri\gamma| = a^4 r^2 \gamma^2 < 4a^4 r^2 < \frac{4a^4}{\Delta\mu^2}\tau^2$$

$$G_{22} = 1 - \gamma^2 r(a^2 r + \theta);$$

θ, r are positive $\Rightarrow G_{22} < 1$

and

$$G_{22} > -\gamma^2 r(a^2 r + \theta) > -\frac{\tau}{\Delta\mu}\left(a^2 \frac{\tau}{\Delta\mu} + \theta\right)$$

so

$$G_{22} \in \left]-\frac{\tau}{\Delta\mu}\left(a^2 \frac{\tau}{\Delta\mu} + \theta\right),\ 1\ \right]$$

We can say also; if we consider the necessary condition which can be written $\frac{\Delta t}{\Delta\mu} < A$; i.e., $r < A$ that all elements of $G(\Delta r, k)$ are bounded for all k which appears in the $\sin k \frac{\Delta\mu}{2}$.

This way can be used for the amplification matrix in Hick's article too.

So the necessary condition III. 10 written strictly is sufficient for the stability of the system.

2.6. Results

The figure 1 presents the value of Δt which must be used when the viscosity goes from 0 to 1500 poises for a mesh of width $\delta x = 0.03$ mm and a $\Delta u = .193$mm/um

We see that the stability condition goes down very fast until 400 poises and more slowly after. Three lines have been drawn. One is the value obtained for CFL condition ($\Delta x/C$). The second is the value of Δt with linear pseudo viscosity. The third is the value given by Thompson (with nonlinear viscosity). We can see that the two first can be not sufficient for stability if a Maxwell's law of behavior is taken and that the last for the Δu used here, can be too restrictive until η be 600 poises.

Figures 2 and 3 show a typical picture that you can get if the stability condition is not or required.

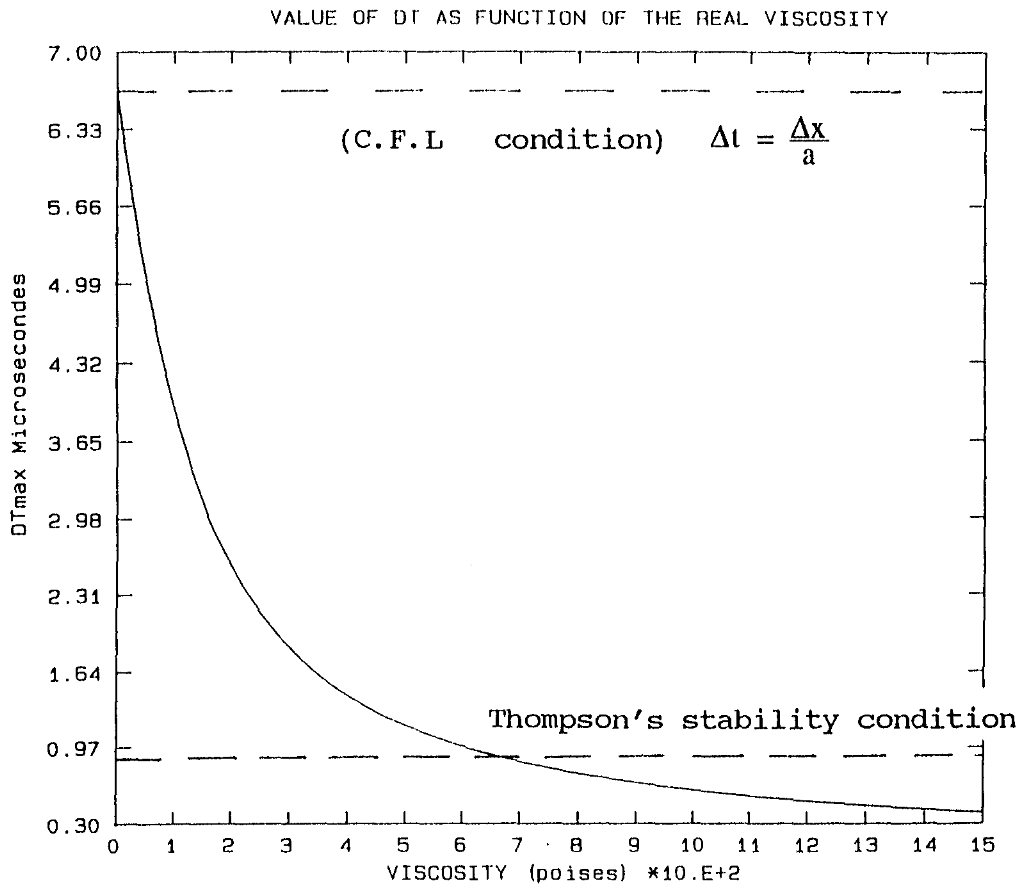


figure 1 : Value of the time step versus the real viscosity for t = .03 mm.

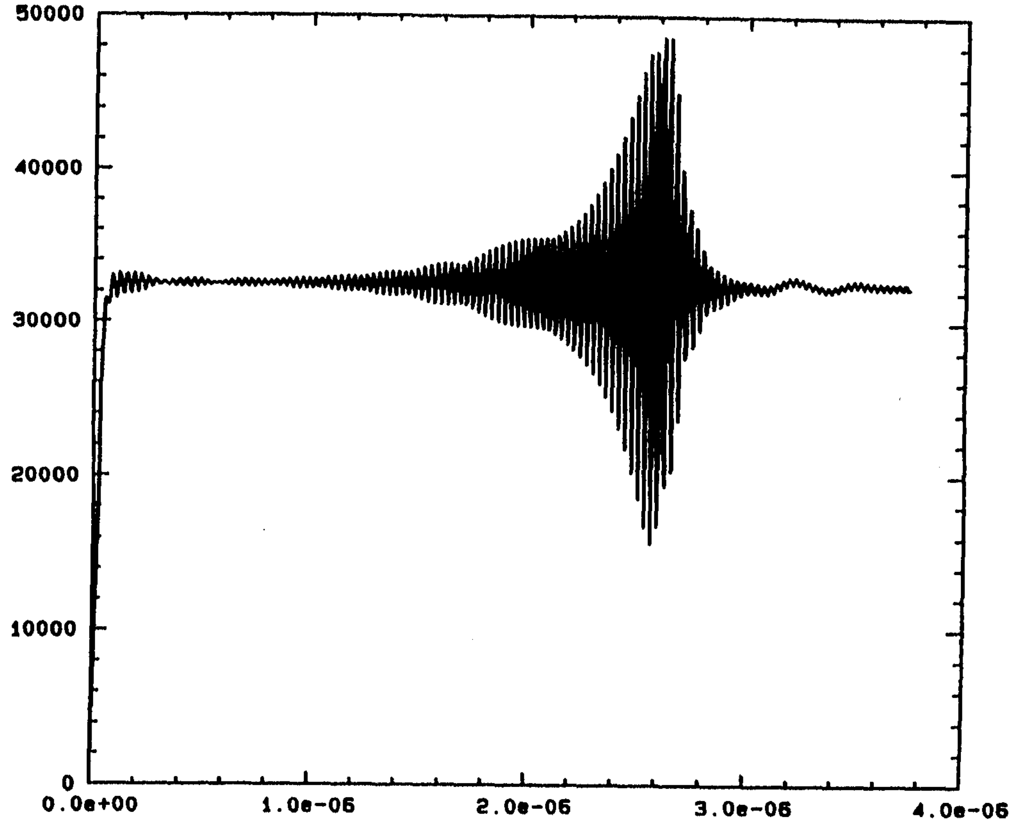


Figure 2 : Signal without stability condition.

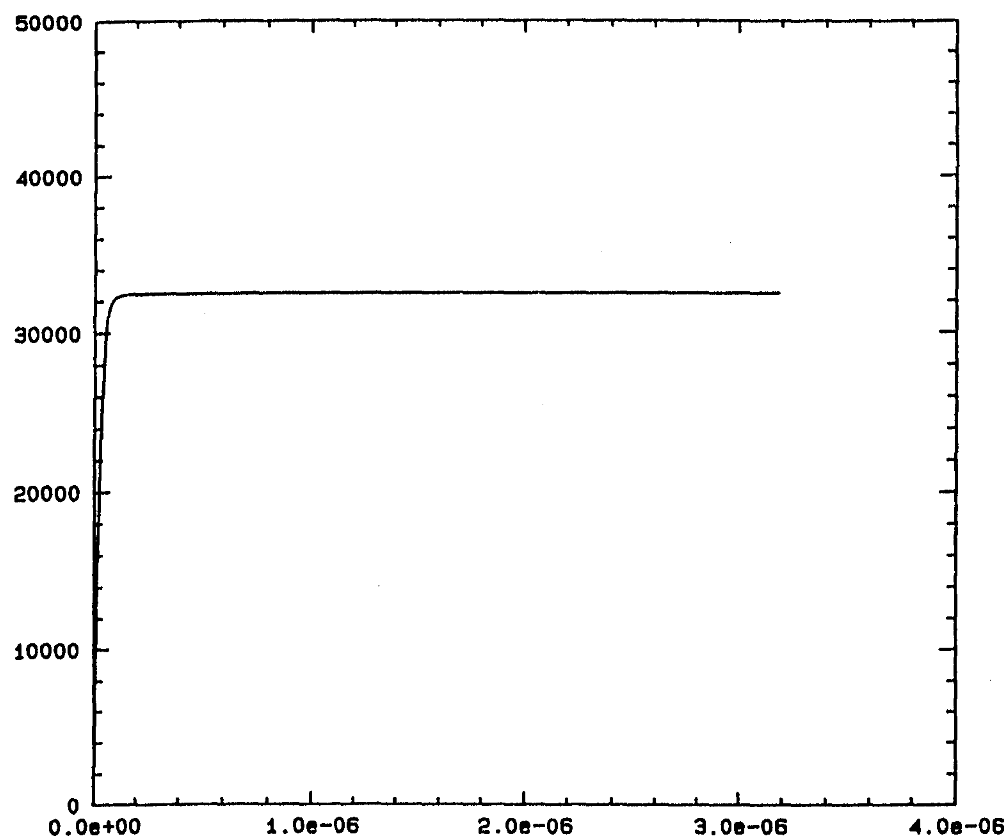


Figure 3 : Signal when stability condition is required

APPENDIX

The differential equation $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$ with I.C $u(x,0) = u_0(x)$ is well posed if $u_0(x)$ belong to L_2 (integrable square function space) and $c>0$.

The classical solution of this equation is given by
$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ik(x-ct)}\,\hat{u}_0(k)\,dk = u_0(x-ct)$$

For all $t \geq 0$, we have:
$$\|u(x,t)\| = \int_{-\infty}^{\infty} u^2(x,t)\,dx = \int_{-\infty}^{\infty} u_0^2(x-ct)\,dx = \int_{-\infty}^{\infty} u_0^2(X)\,dX$$
$$= \|u_0\|$$

So the problem is well posed with $K=1$ and $\mu = 0$ (theorem chp I)

BIBLIOGRAPHY

(1) H. Ziegler, An Introduction to Thermomechanics, North-Holland Pub. Co. 1977.

(2) R. D. Richtmeyer, K. W. Morton, Different Methods for Initial Value Problems 2nd ed., Intersience Pub. (1967).

(3) Heinz-Otto Kreiss, Difference Approximations for Time Dependent Problems. AGARD Lecture Series **73** (1975) p. 4.1 - 4.33.

(4) K. W. Morton, Stability and Accuracy of Numerical Approximations to Time Dependent Flows, AGARD Lecture Series 73 (1975).

(5) D. L. Hicks, Stability Analysis of Wondy for a Special Case.of Maxwell's Law. Mathematics of Computation, V. 32, number 144, Oct 1978, p. 1123-1130.

(6) R. Peyret-Taylor, Computational Methods for Fluid Flow, Springer-Verlag (1983).

(7) See the works of P. D. Lax, and A. Harten.

(8) W. Herrmann, P. Holzhauser, R. J. Thompson, Sandia Laboratory, SC. RR. 66, 601, Feb 1967.

INTERNAL REPORTS - 1992

1. K.G. Casey, "Absorption Spectroscopy at SDL", Internal Report 92-01, June 1992.

2. K.G. Casey, "Absorption Shot Experiments: NITROMETHANE", Internal Report 92-02, June 1992.