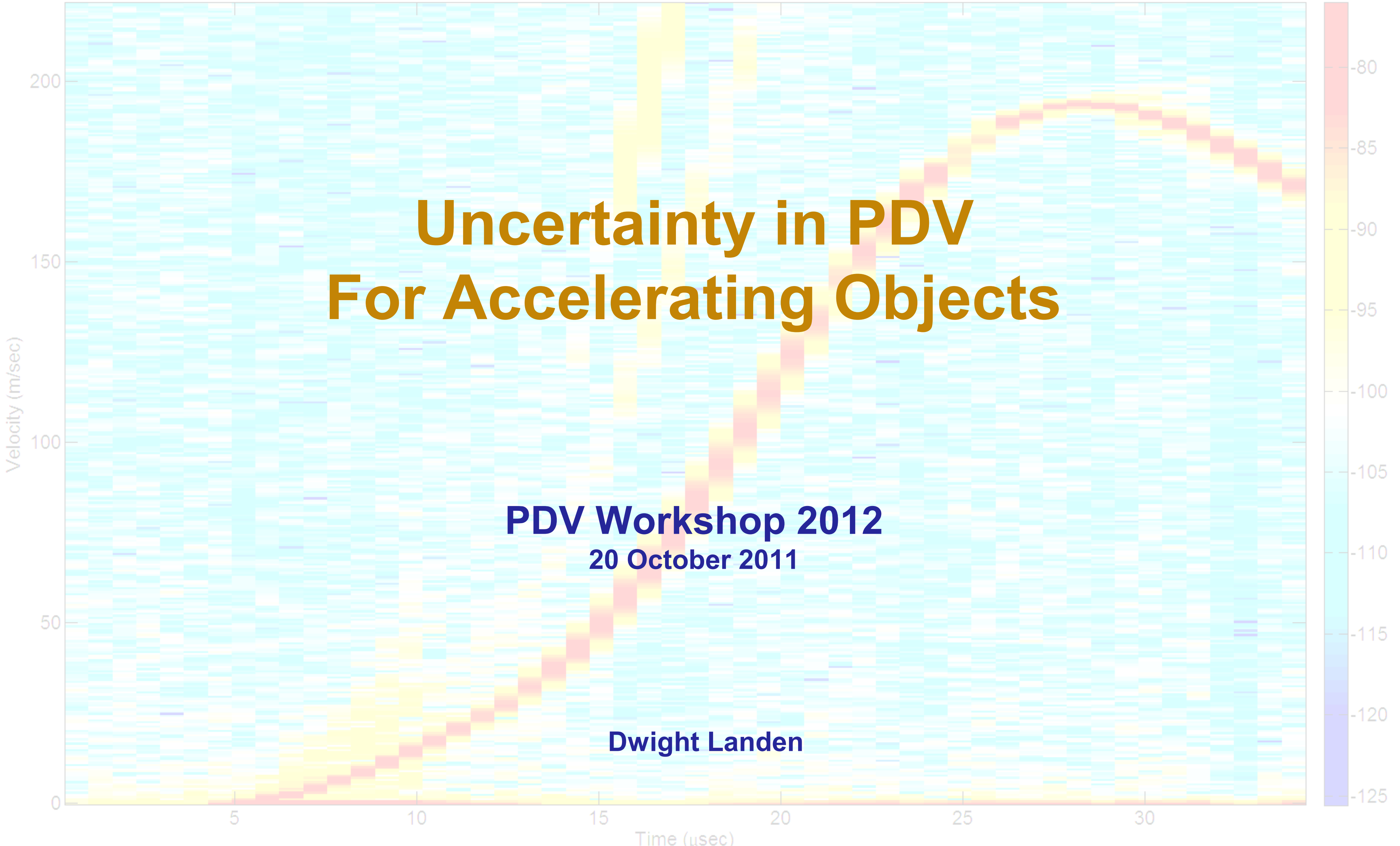

Uncertainty in PDV For Accelerating Objects

PDV Workshop 2012

20 October 2011

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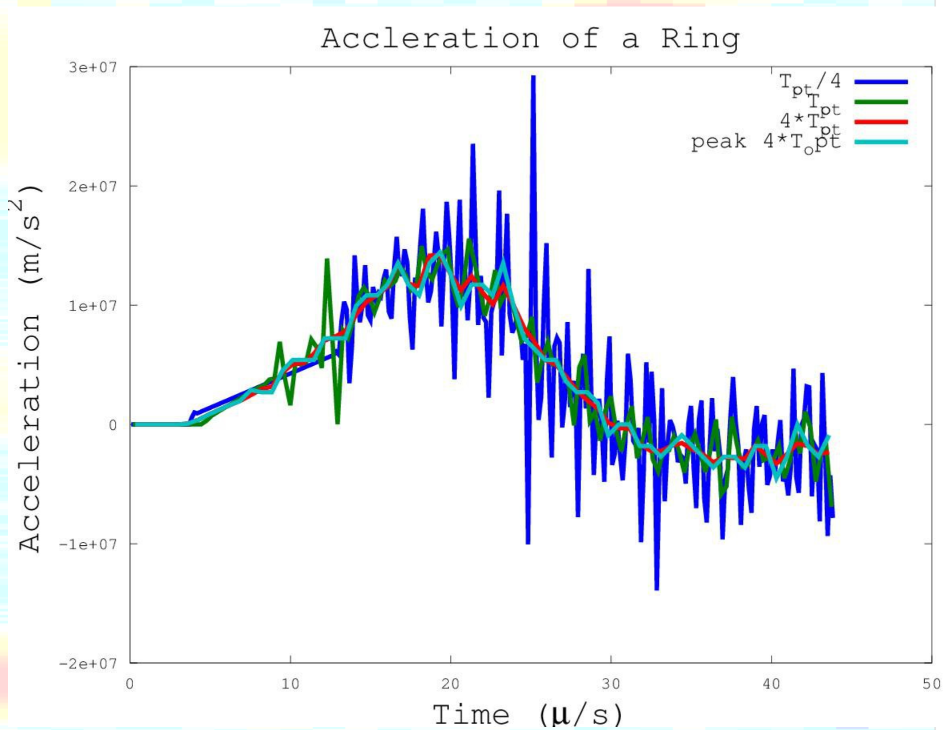
Overview

- **Background**
  - Interferometry
  - Basic Assumptions
- **Time Frequency Analysis**
  - Linear velocity
  - Non-Linear Velocity
  - Moments
  - Expansions
  - Is Skewness a problem
  - How Big is the Error?
  - Optimal Window Size
  - Accuracy Error
- **Noise**
  - Noise in the Spectrogram
  - Noise Analysis
  - Uncertainty in Local Mean Frequency
  - Uncertainty and Window Size
  - Uncertainty with Thresholding
- **Conclusion**

Background

- Trying to justify window size in expanding ring experiment
- Was I cheating by increasing the window size?
- What else is happening to the distribution?
- Derived “optimal window” for velocity
- Looking for “optimal window” for acceleration
- Stumbled onto Cohen's work



Interferometry

- **Fields of a light wave**
  - The E field amplitude of a wave traveling in the z direction

    $$E(t) = E_1 \sin(\omega_1 t)\boldsymbol{i} + E_1 \cos(\omega_1 t)\boldsymbol{j}$$

  - Orthogonal B field

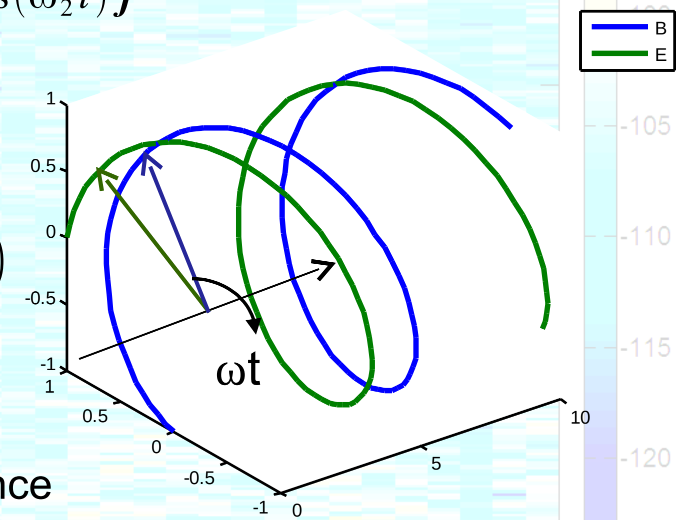
    $$B(t) = -B_1 \cos(\omega_1 t)\boldsymbol{i} + B_1 \sin(\omega_1 t)\boldsymbol{j}$$

- **Fields add**

  $$E(t) = E_1 \sin(\omega_1 t)\boldsymbol{i} + E_2 \sin(\omega_2 t)\boldsymbol{i} + E_1 \cos(\omega_1 t)\boldsymbol{j} + E_2 \cos(\omega_2 t)\boldsymbol{j}$$

- **Light intensity is a square law**
  - From Poynting and Maxwell

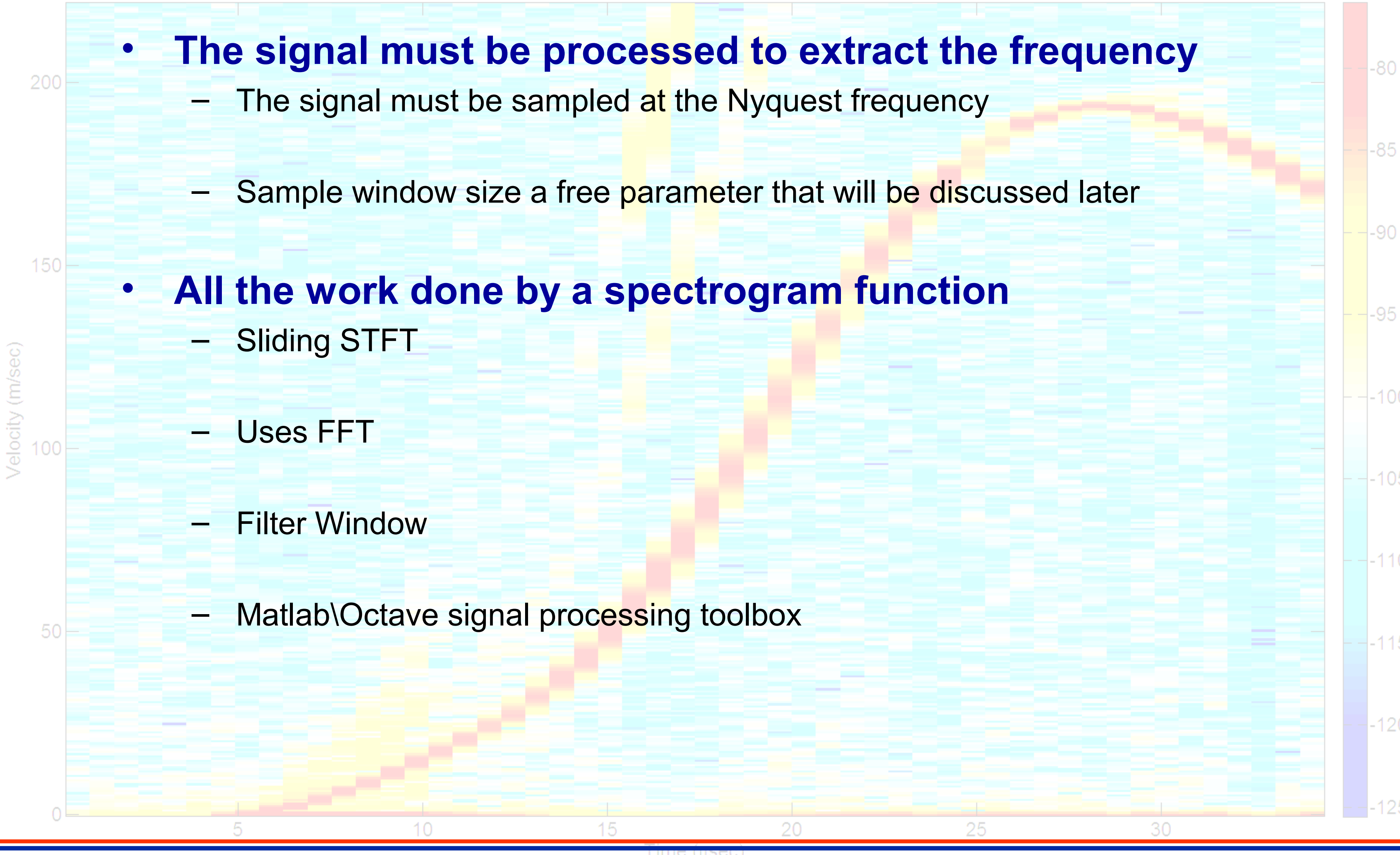
$$S(t) = \frac{1}{\mu_0 c}\left(E_1^2(t) + E_2^2(t) + 2E_1 E_2 \cos(\omega_1 t - \omega_2 t)\right)$$



  - This equation is proportional to intensity
  - The third term is related to the frequency difference in two signals

Basic Assumptions

- **The signal must be processed to extract the frequency**
  - The signal must be sampled at the Nyquest frequency
  - Sample window size a free parameter that will be discussed later
- **All the work done by a spectrogram function**
  - Sliding STFT
  - Uses FFT
  - Filter Window
  - Matlab\Octave signal processing toolbox



Time Frequency Analysis

- **Signal** $d(t) = A(t)\,e^{i\phi(t)}; \quad D(\omega) = \frac{1}{\sqrt{(2\pi)}}\int d(t)\,e^{-i\omega t}\,dt$

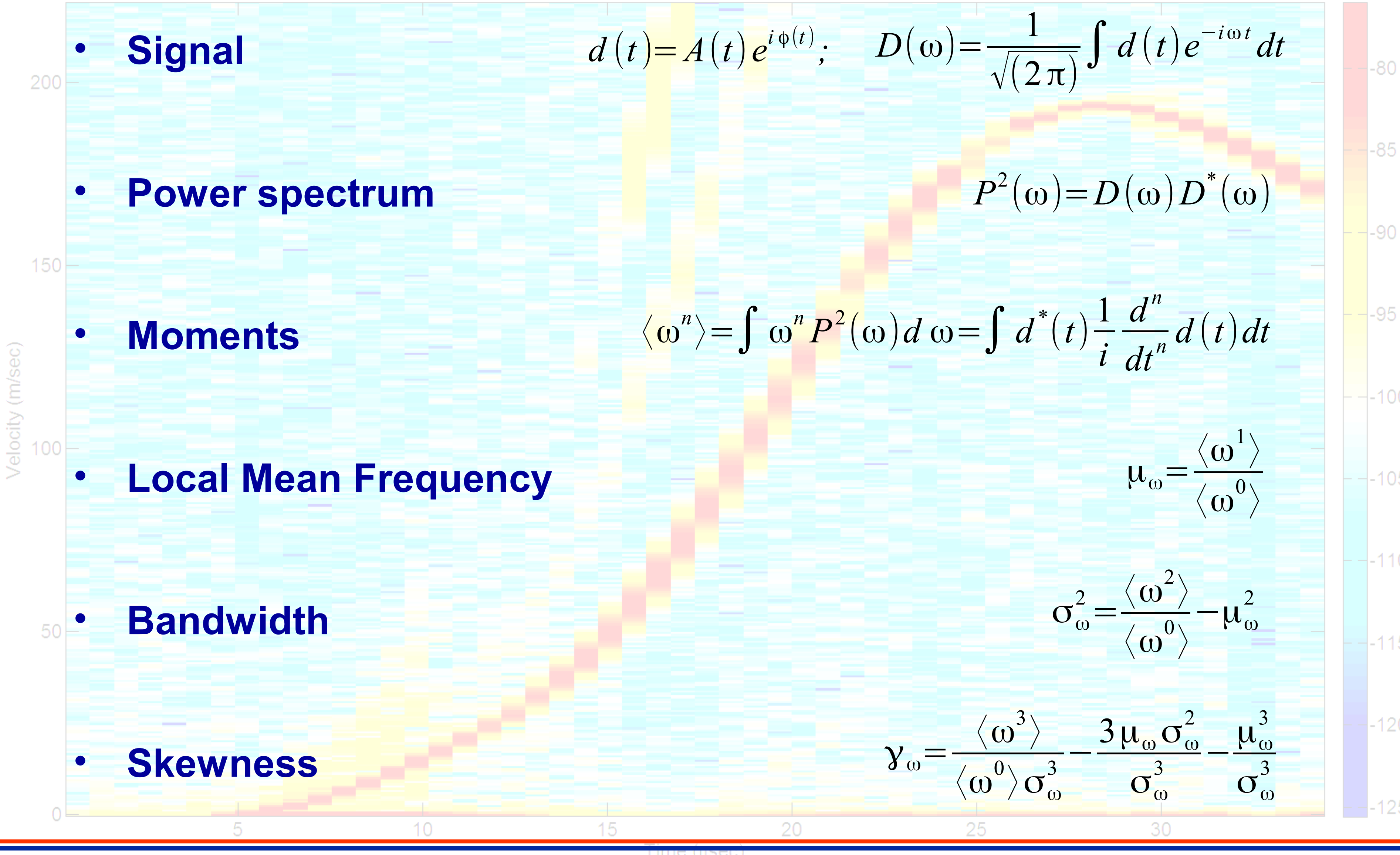
- **Power spectrum** $P^2(\omega) = D(\omega)\,D^*(\omega)$

- **Moments** $\langle \omega^n \rangle = \int \omega^n P^2(\omega)\,d\omega = \int d^*(t)\frac{1}{i}\frac{d^n}{dt^n}d(t)\,dt$

- **Local Mean Frequency** $\mu_\omega = \frac{\langle \omega^1 \rangle}{\langle \omega^0 \rangle}$

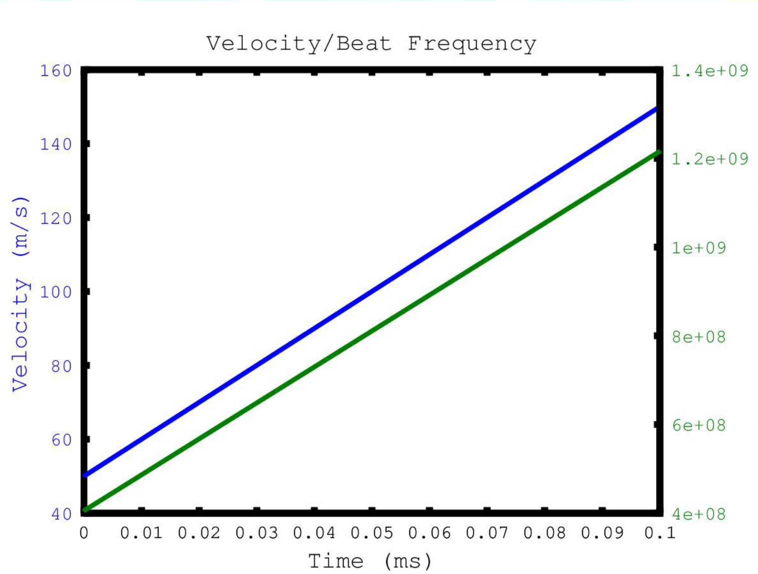
- **Bandwidth** $\sigma_\omega^2 = \frac{\langle \omega^2 \rangle}{\langle \omega^0 \rangle} - \mu_\omega^2$

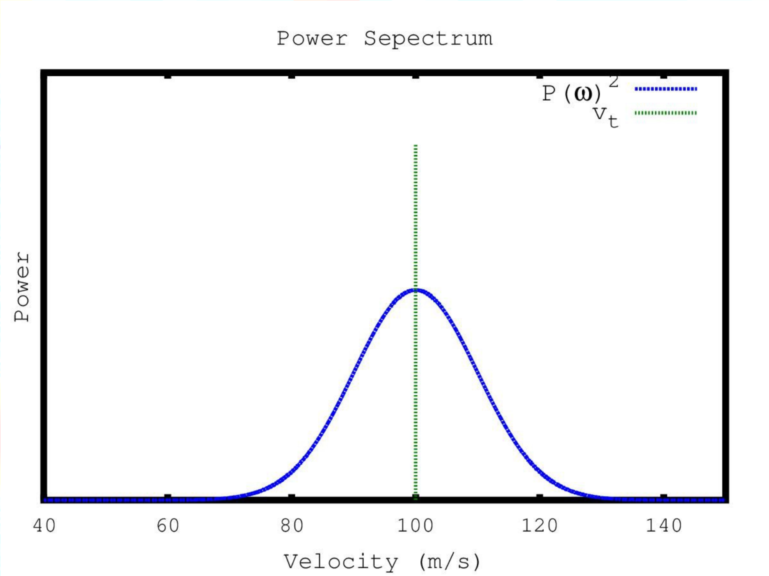
- **Skewness** $\gamma_\omega = \frac{\langle \omega^3 \rangle}{\langle \omega^0 \rangle \sigma_\omega^3} - \frac{3\mu_\omega \sigma_\omega^2}{\sigma_\omega^3} - \frac{\mu_\omega^3}{\sigma_\omega^3}$



Linear velocity

- **Linear velocity + Gaussian window = Gaussian power spectrum**
- **Peak, mean, and velocities line up**





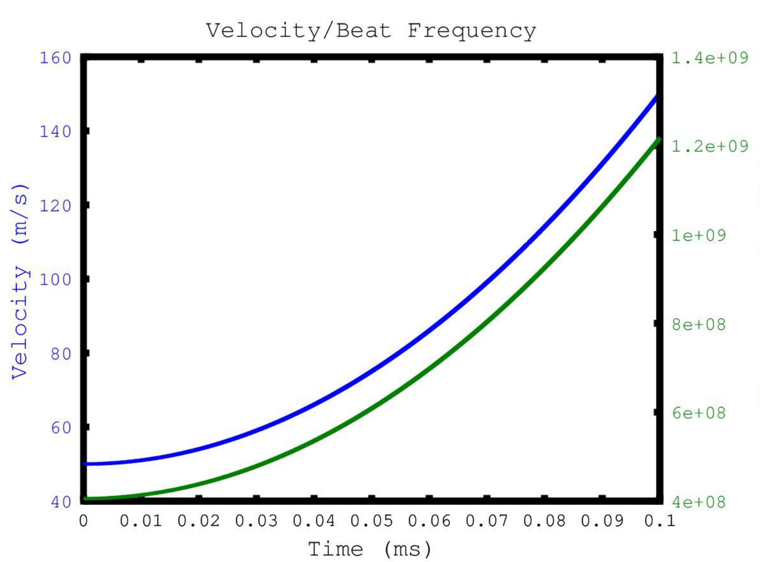
$$v(t) = v_0 + a\,t$$

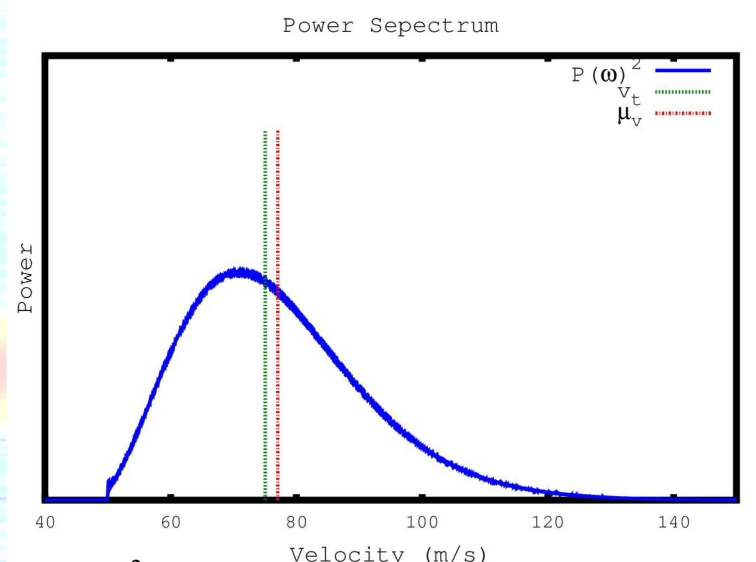
$$\phi(t) = \frac{4\pi}{\lambda_0}\left(v_0 t + \frac{a_0}{2} t^2 + \phi_0\right)$$

$$P(v)^2 = \frac{\lambda_0}{8}\sqrt{\frac{\sigma^2}{2\sigma_v^2}}\, e^{-\frac{(v-\bar{v})}{2\sigma_v^2}}$$

Non-Linear Velocity

- **Non-Linear Velocity → ?**
- **Spectrum is Skewed**
- **Peak, mean, and velocity no longer line up**





$$v(t)=v_0+a_0t+j_0t^2+s_0t^3+\ldots$$

$$\phi(t)=\frac{4\pi}{\lambda_0}\left(v_0t+\frac{a_0}{2}t^2+\frac{j_0}{3}t^3+\frac{s_0}{4}t^4+\ldots\right)$$

$$P(\omega)^2=¿$$

$$arg\left(max\left(P(\omega)^2\right)\right)=¿$$

$$\mu_\omega=\dot{\phi}(\bar{t})+\frac{\sigma^2}{2}\dddot{\phi}(\bar{t})+\frac{\sigma^4}{32}\sigma^{(5)}+\ldots$$

Moments

- **Zero'th**

$$\langle \omega_0 \rangle = |P|^2 = \int A^2(t)\,dt$$

- **First**

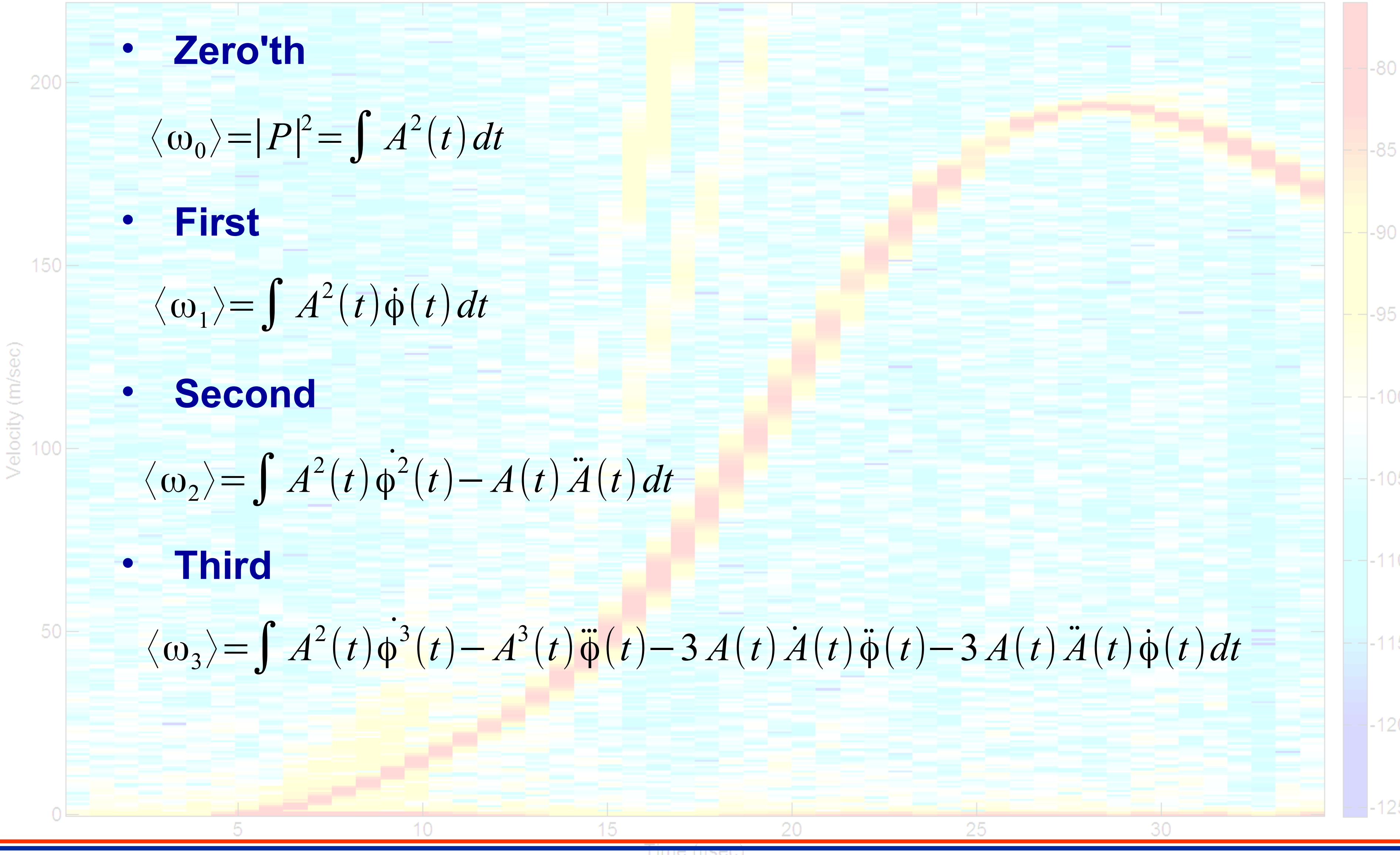
$$\langle \omega_1 \rangle = \int A^2(t)\,\dot{\phi}(t)\,dt$$

- **Second**

$$\langle \omega_2 \rangle = \int A^2(t)\,\dot{\phi}^2(t) - A(t)\,\ddot{A}(t)\,dt$$

- **Third**

$$\langle \omega_3 \rangle = \int A^2(t)\,\dot{\phi}^3(t) - A^3(t)\,\dddot{\phi}(t) - 3\,A(t)\,\dot{A}(t)\,\ddot{\phi}(t) - 3\,A(t)\,\ddot{A}(t)\,\dot{\phi}(t)\,dt$$



Expansions

- **Local Mean Frequency**

$$\mu_\omega = \dot{\phi}(\bar{t}) + \frac{\sigma^2}{2}\ddot{\phi}(\bar{t}) + \frac{\sigma^4}{32}\sigma^{(5)} + \ldots$$

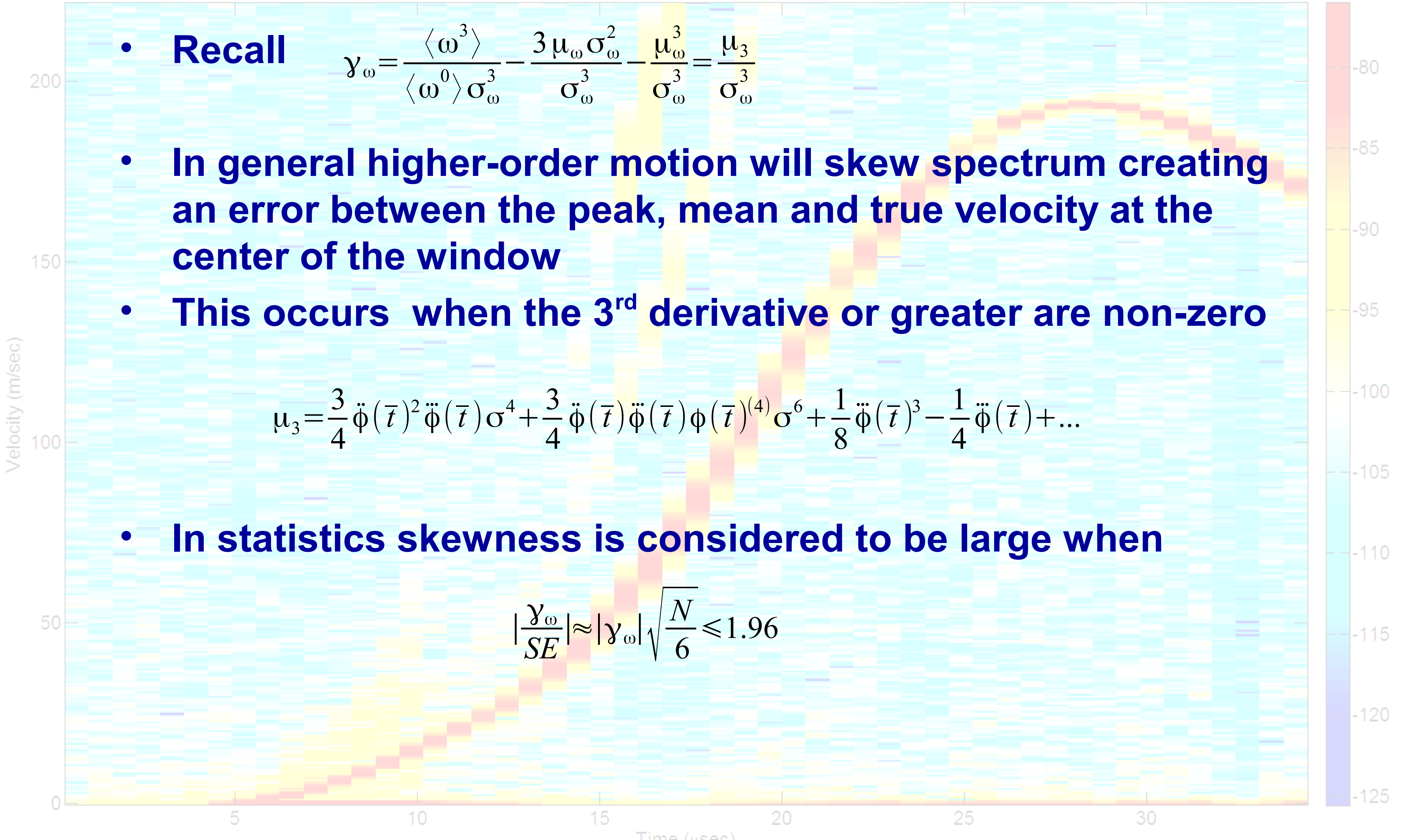
- **Bandwidth**

$$\sigma_\omega^2 = \frac{1}{2\sigma^2} + \frac{\sigma^2}{2}\ddot{\phi}(\bar{t})^2 + \frac{\sigma^4}{4}[\ddot{\phi}(\bar{t})\phi^{(4)}(\bar{t}) + \frac{1}{2}\dddot{\phi}(\bar{t})^2] + \ldots$$

- **Skewness**

$$\gamma_\omega = \frac{1}{\sigma_\omega^3}(\frac{3}{4}\ddot{\phi}(\bar{t})^2\dddot{\phi}(\bar{t})\sigma^4 + \frac{3}{4}\ddot{\phi}(\bar{t})\dddot{\phi}(\bar{t})\phi(\bar{t})^{(4)}\sigma^6 + \frac{1}{8}\dddot{\phi}(\bar{t})^3 - \frac{1}{4}\dddot{\phi}(\bar{t}) + \ldots)$$

Is Skewness a problem



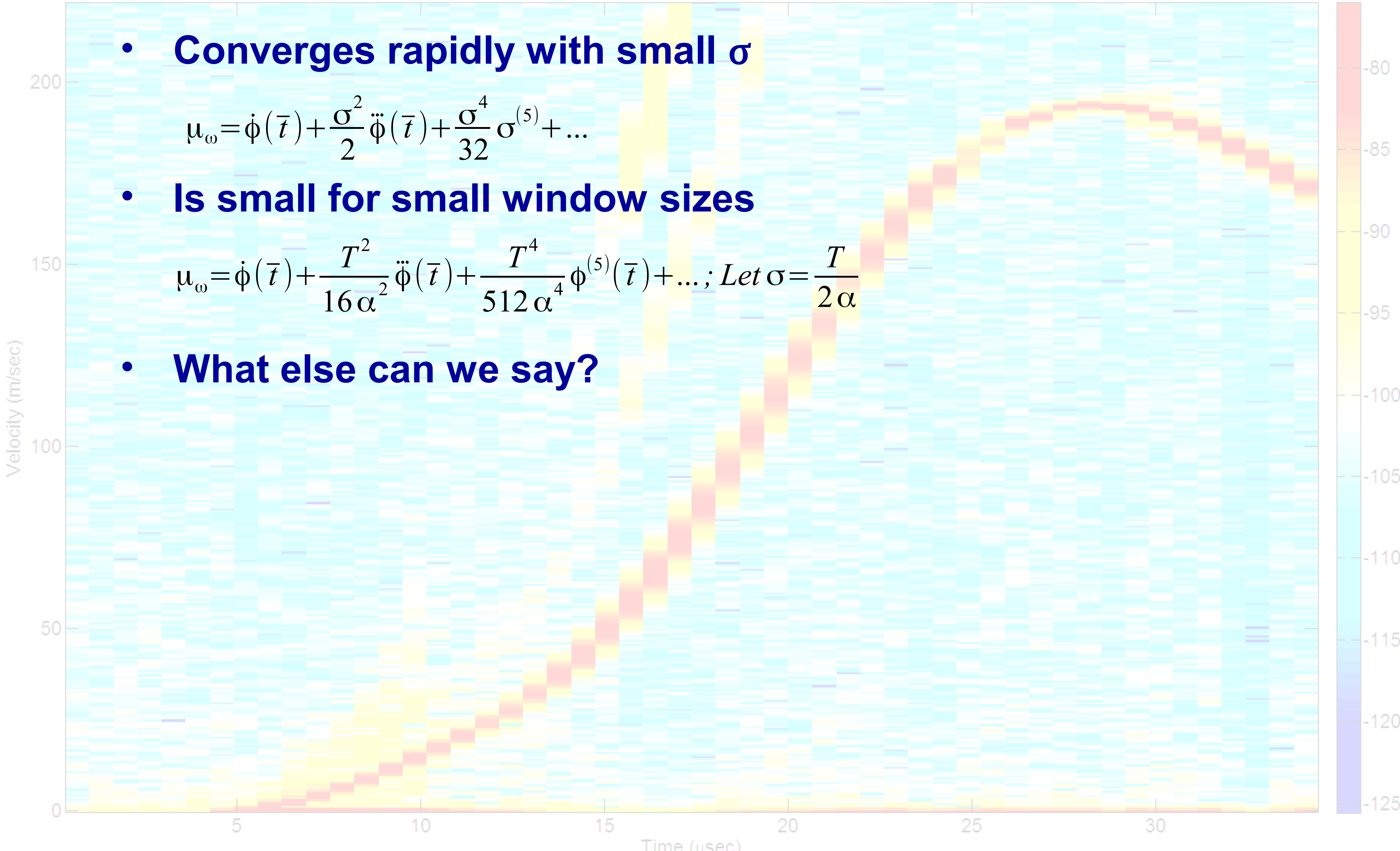
- **Recall** $$\gamma_\omega = \frac{\langle \omega^3 \rangle}{\langle \omega^0 \rangle \sigma_\omega^3} - \frac{3\mu_\omega \sigma_\omega^2}{\sigma_\omega^3} - \frac{\mu_\omega^3}{\sigma_\omega^3} = \frac{\mu_3}{\sigma_\omega^3}$$
- **In general higher-order motion will skew spectrum creating an error between the peak, mean and true velocity at the center of the window**
- **This occurs when the 3rd derivative or greater are non-zero**

$$\mu_3 = \frac{3}{4}\ddot{\phi}(\bar{t})^2 \dddot{\phi}(\bar{t})\sigma^4 + \frac{3}{4}\ddot{\phi}(\bar{t})\dddot{\phi}(\bar{t})\phi(\bar{t})^{(4)}\sigma^6 + \frac{1}{8}\dddot{\phi}(\bar{t})^3 - \frac{1}{4}\dddot{\phi}(\bar{t}) + \ldots$$

- **In statistics skewness is considered to be large when**

$$\left|\frac{\gamma_\omega}{SE}\right| \approx |\gamma_\omega|\sqrt{\frac{N}{6}} \leqslant 1.96$$

How Big is the Error?



- **Converges rapidly with small σ**

$$\mu_\omega = \dot{\phi}(\bar{t}) + \frac{\sigma^2}{2}\ddot{\phi}(\bar{t}) + \frac{\sigma^4}{32}\sigma^{(5)} + \ldots$$

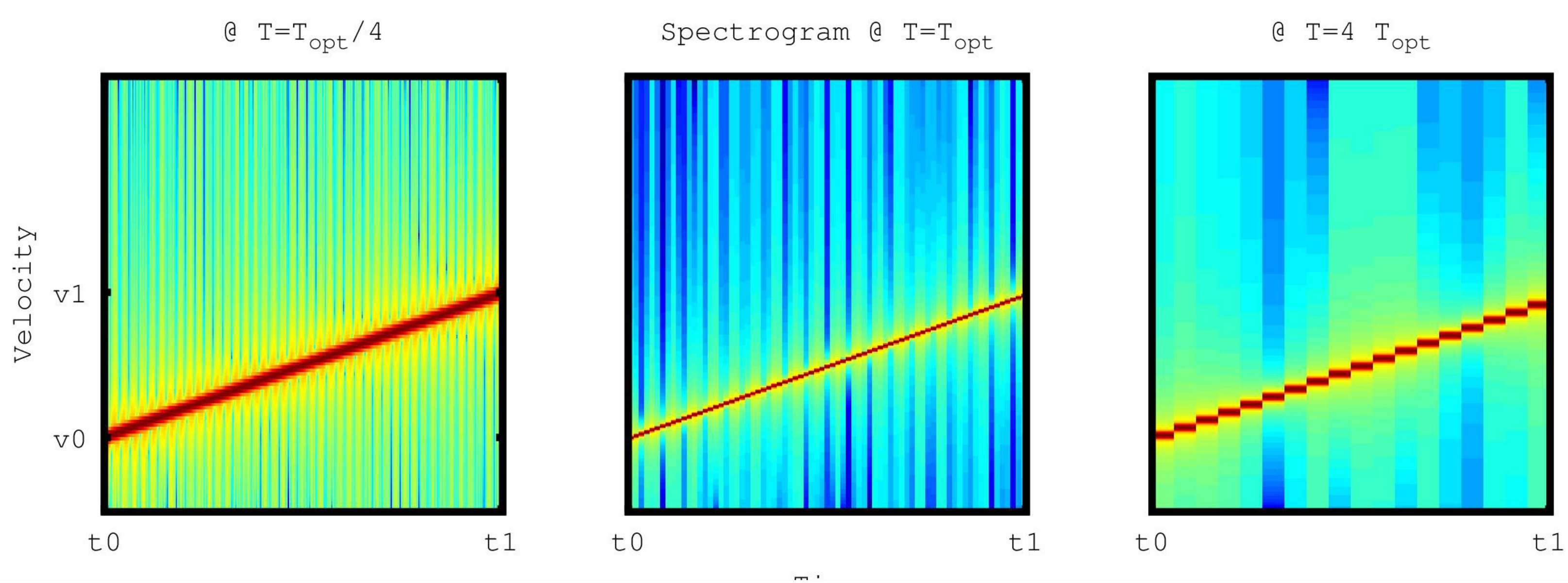
- **Is small for small window sizes**

$$\mu_\omega = \dot{\phi}(\bar{t}) + \frac{T^2}{16\alpha^2}\dddot{\phi}(\bar{t}) + \frac{T^4}{512\alpha^4}\phi^{(5)}(\bar{t}) + \ldots;\ Let\ \sigma = \frac{T}{2\alpha}$$

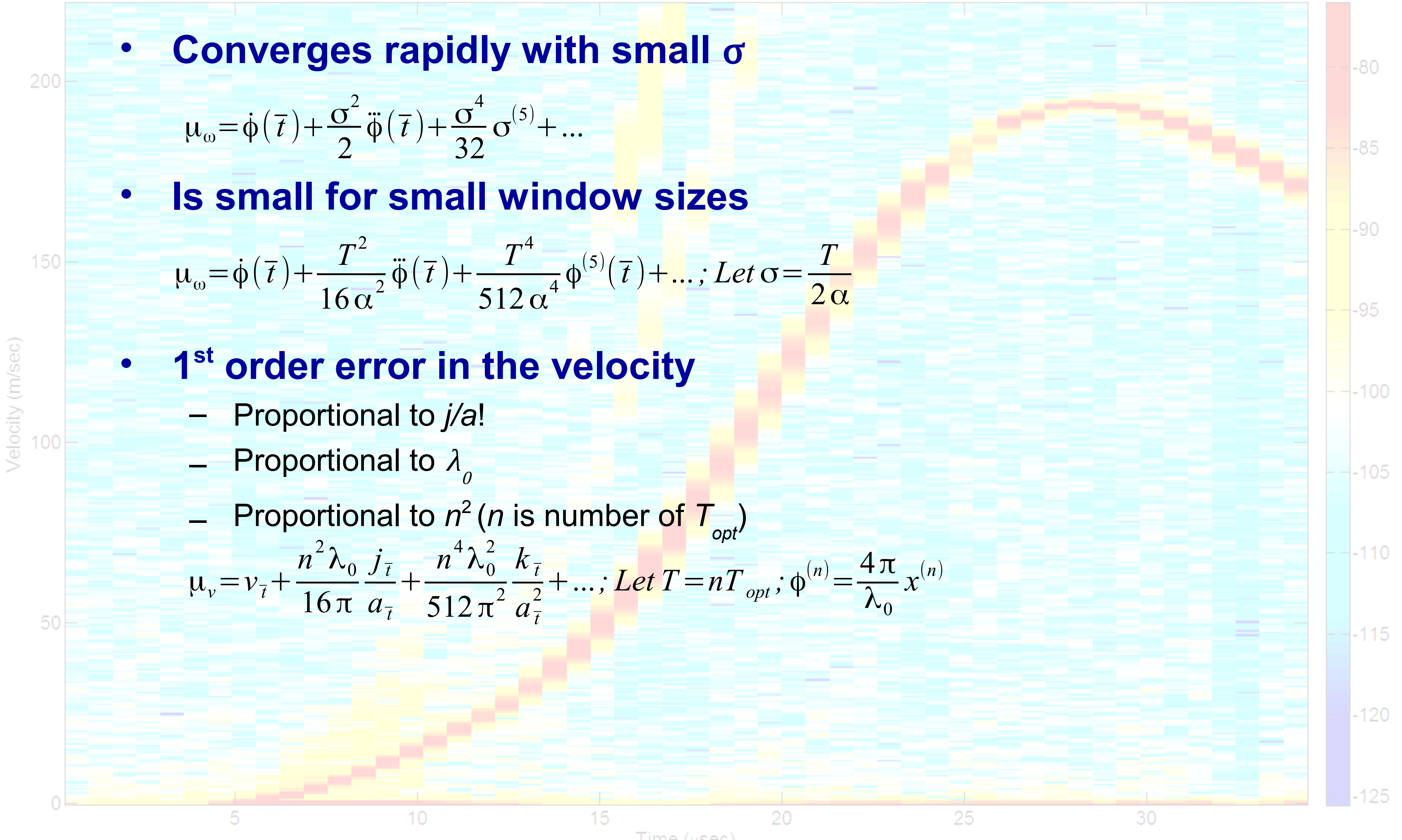
- **What else can we say?**

Optimal Window Size

- **The minimum bandwidth window is** $T_{opt} = \sqrt{\dfrac{\lambda_0}{\pi a_{\bar{t}}}}\,\alpha$
- **Smaller windows will reduce error in accuracy, however the spectrum will be spread out and have a low resolution resulting in integration error**
- **Larger windows can increase resolution but risk accuracy**



Accuracy Error



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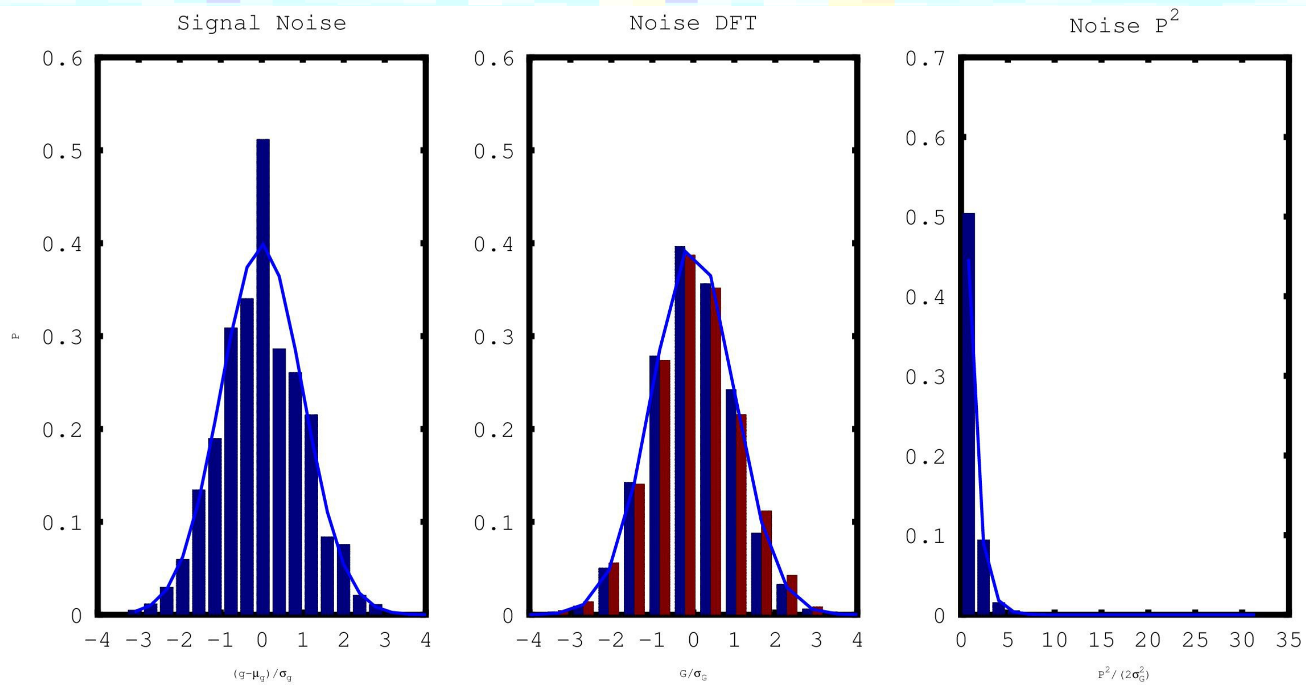
- **1st order error in the velocity**
  - Proportional to *j/a*!
  - Proportional to λ_0
  - Proportional to n^2 (*n* is number of T_{opt})

$$\mu_v = v_{\bar{t}} + \frac{n^2\lambda_0}{16\pi}\frac{j_{\bar{t}}}{a_{\bar{t}}} + \frac{n^4\lambda_0^2}{512\pi^2}\frac{k_{\bar{t}}}{a_{\bar{t}}^2} + \ldots;\ Let\ T = nT_{opt};\ \phi^{(n)} = \frac{4\pi}{\lambda_0}x^{(n)}$$

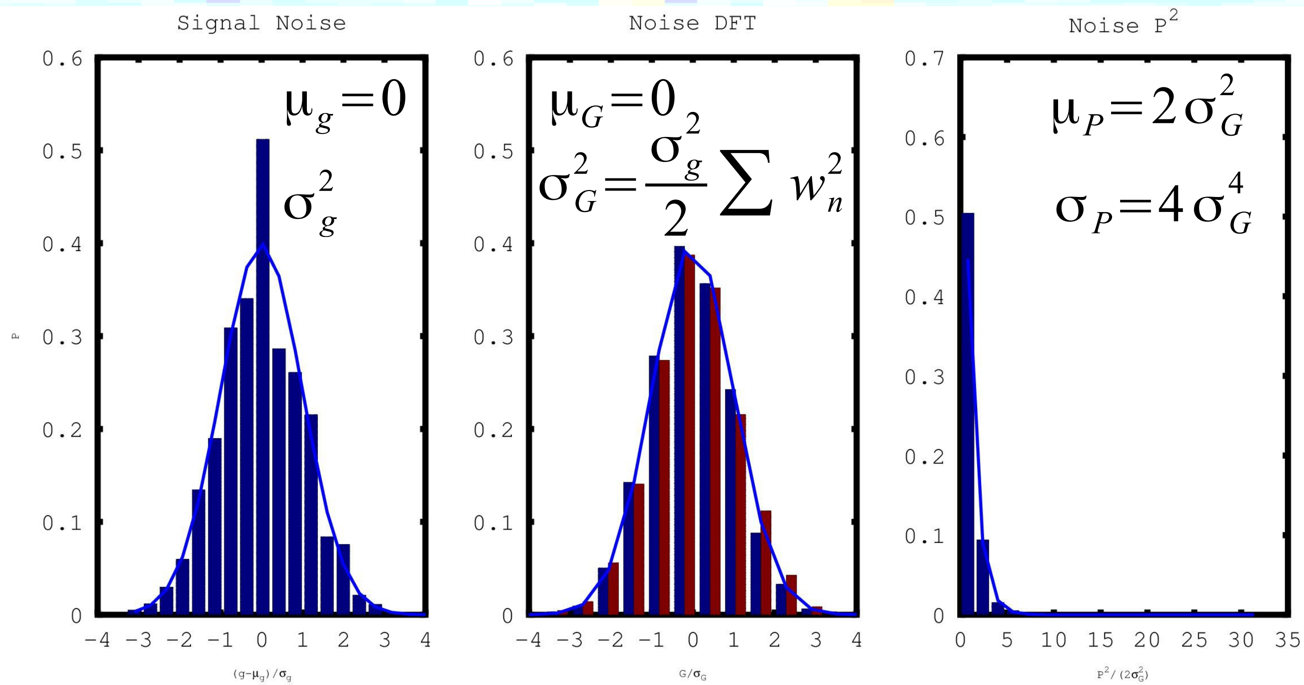
Noise

- **Noise has multiple components**
  - Shot noise → Poisson statistic
  - Thermal → Gauss statistic
- **Gaussian is good approximation and is standard practice in detector design.**

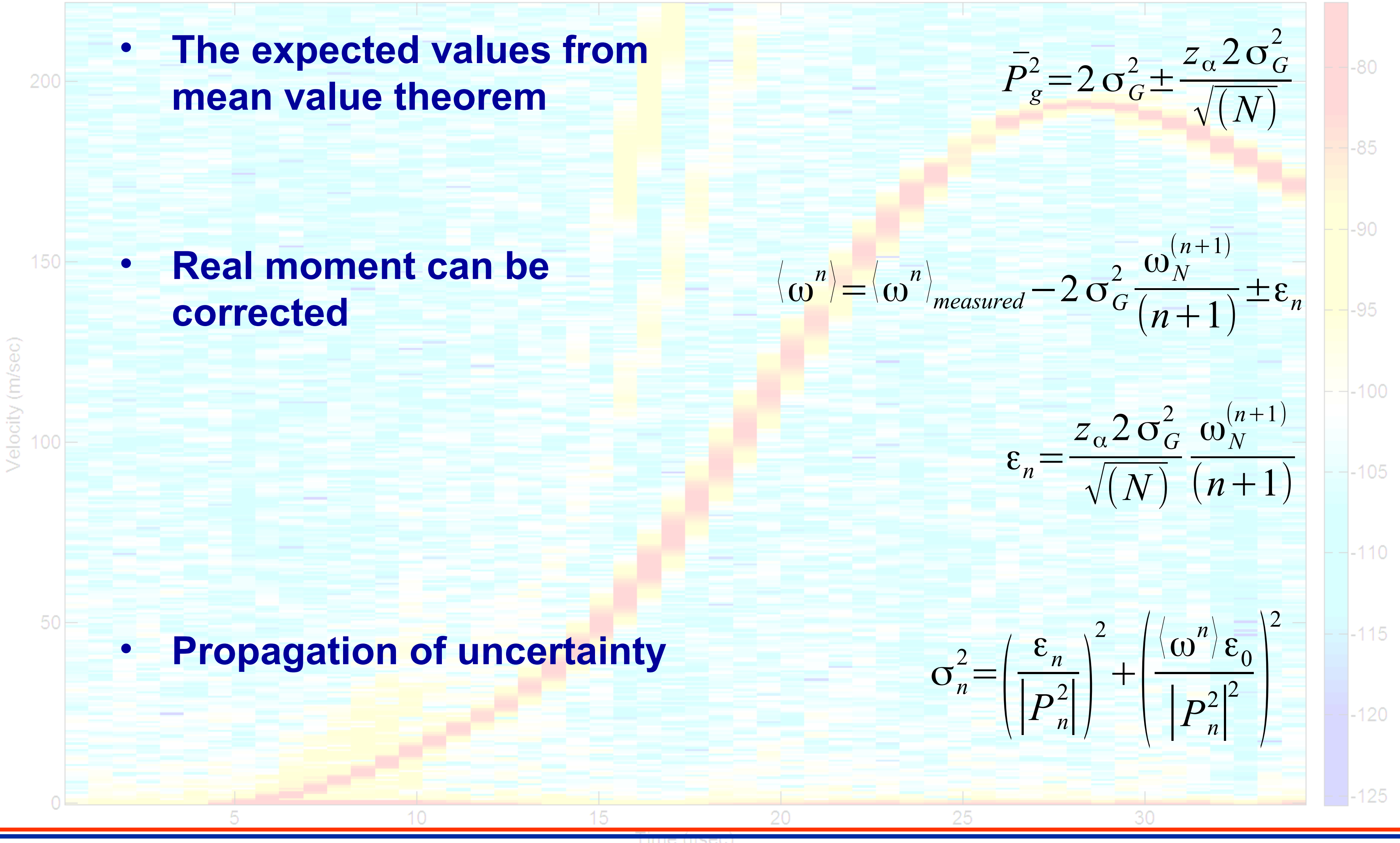


Noise in the Spectrogram

- When processed by the STFT Gaussian distributed noise sample become Gaussian distributed Fourier coefficients
- The power spectrum follows the exponential distribution
- The mean value of the power spectrum is normally distributed



Noise Analysis



- The expected values from mean value theorem

$$\bar{P}_g^2 = 2\sigma_G^2 \pm \frac{z_\alpha 2\sigma_G^2}{\sqrt{(N)}}$$

- Real moment can be corrected

$$\langle\omega^n\rangle = \langle\omega^n\rangle_{measured} - 2\sigma_G^2 \frac{\omega_N^{(n+1)}}{(n+1)} \pm \varepsilon_n$$

$$\varepsilon_n = \frac{z_\alpha 2\sigma_G^2}{\sqrt{(N)}} \frac{\omega_N^{(n+1)}}{(n+1)}$$

- Propagation of uncertainty

$$\sigma_n^2 = \left(\frac{\varepsilon_n}{\left|P_n^2\right|}\right)^2 + \left(\frac{\langle\omega^n\rangle \varepsilon_0}{\left|P_n^2\right|^2}\right)^2$$

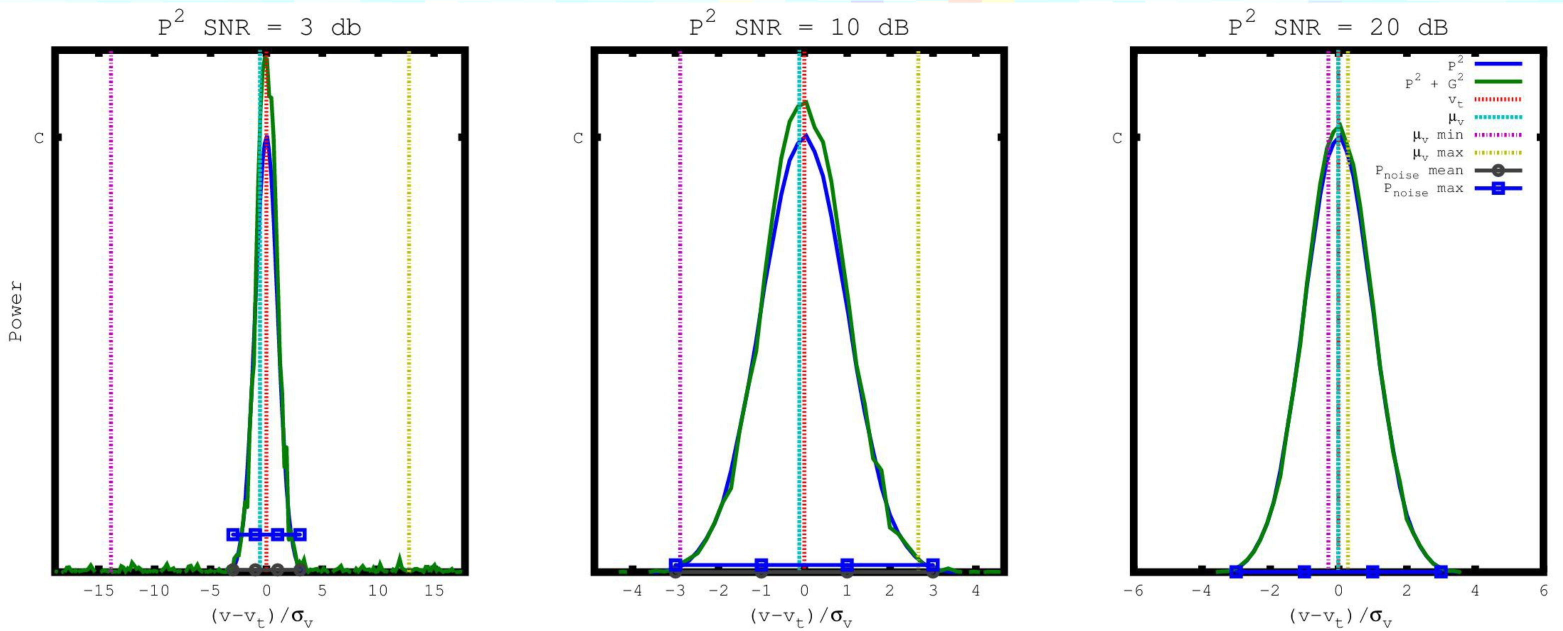
Uncertainty in Local Mean Frequency

- **Noise corrected local mean frequency with uncertainty**

$$\bar{\mu}_\omega = \frac{\langle\omega\rangle_{measured} - \sigma_G^2\,\omega_N^2}{\left|P_m^2\right| - 2\,\sigma_G^2\,\omega_N} \pm \sigma_1$$

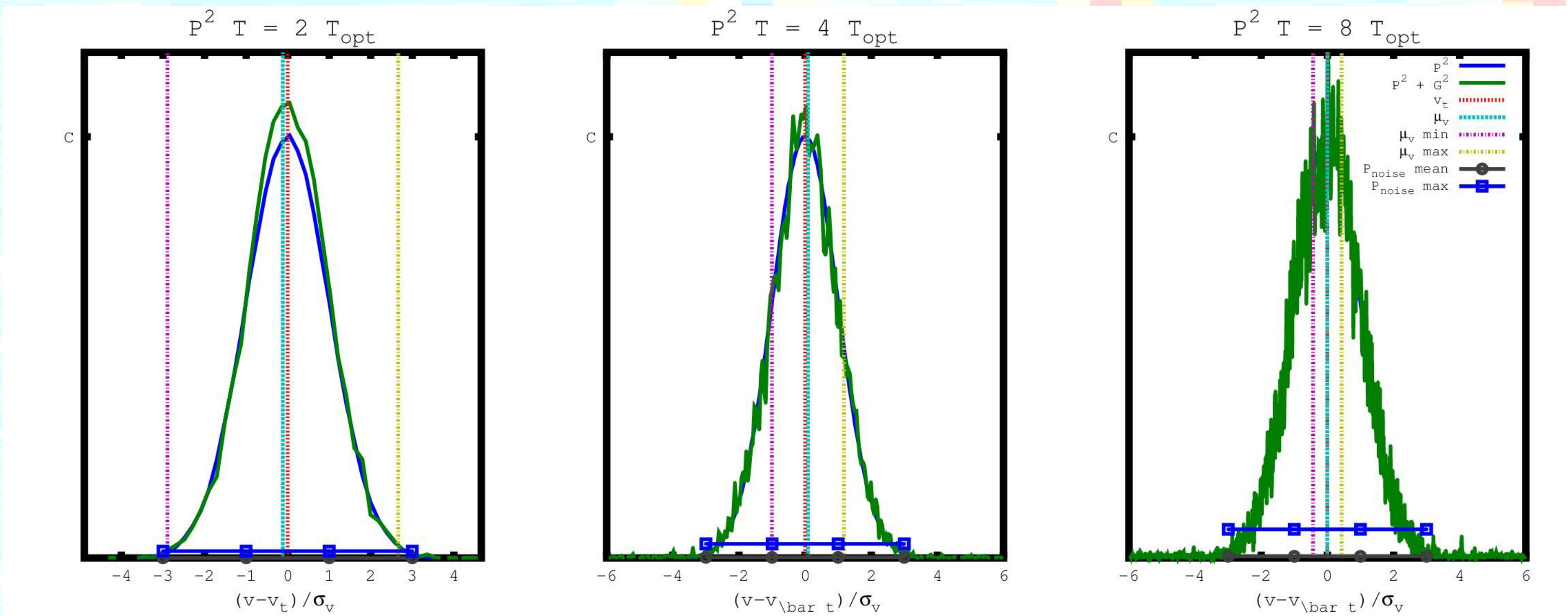
- **Can be quite large when integrating over all w**

  In practice always threshold and integrate only over the peak



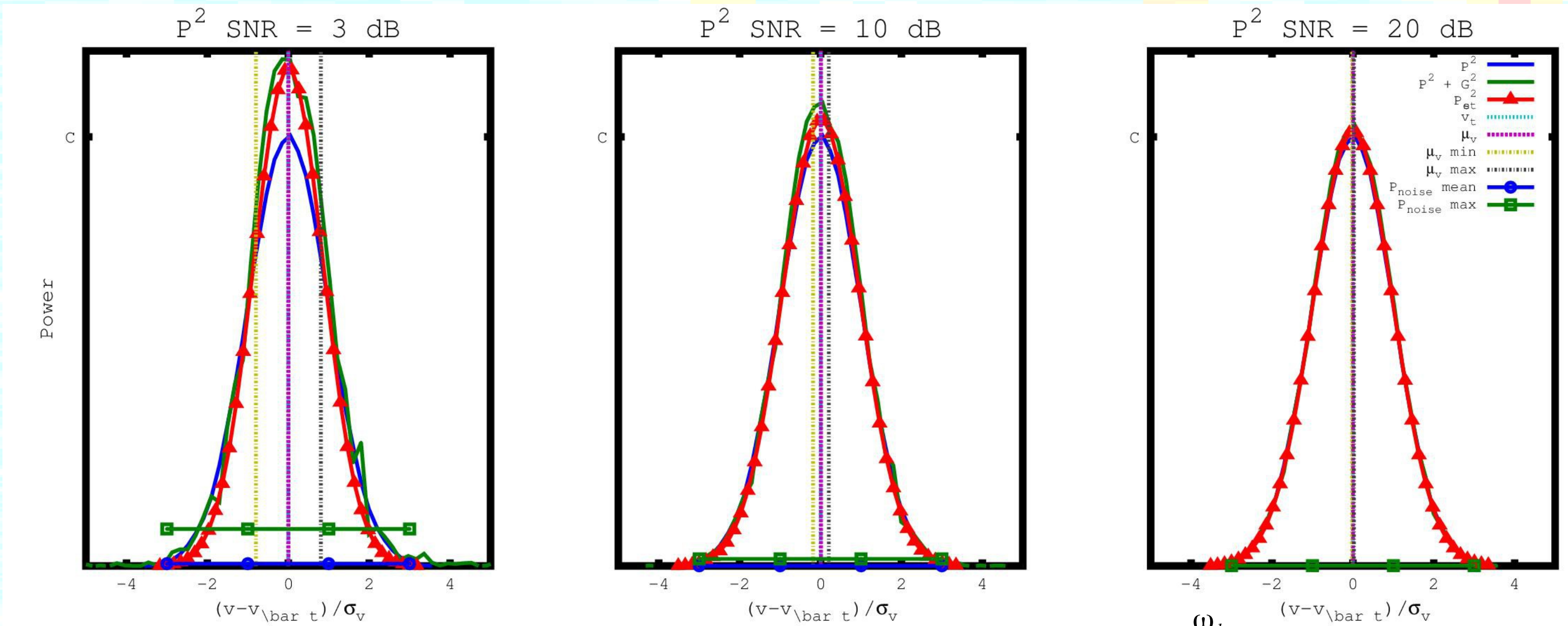
Uncertainty and Window Size

- **At 10 dB for different window sizes**



Uncertainty with Thresholding

- **Performing the integration only over the peak reduces noise error**



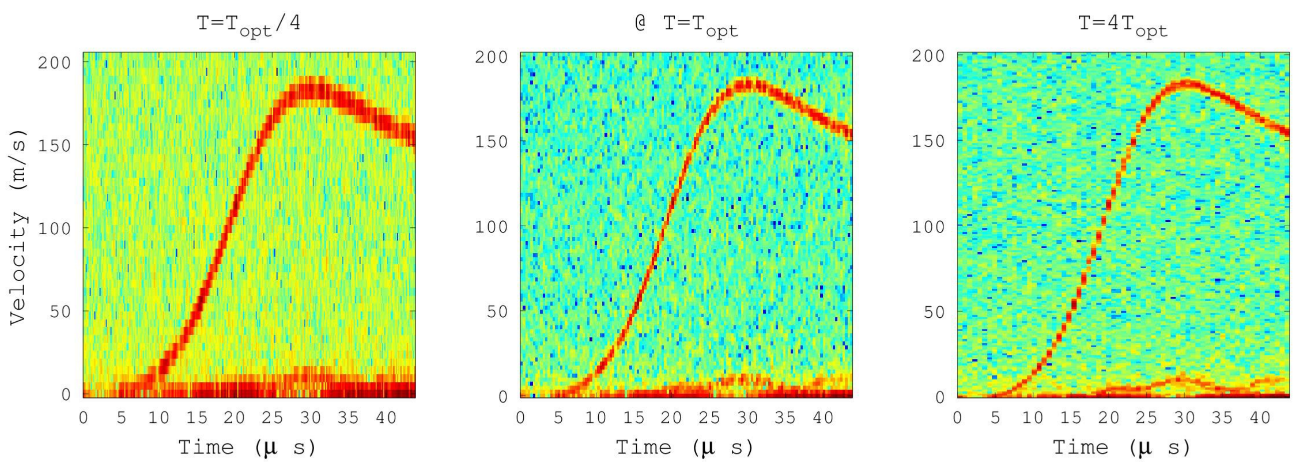
$$\varepsilon_n = \frac{z_\alpha 2\sigma_G^2}{\sqrt{(N)}} \sum_{\omega_a}^{\omega_b} (q_n \omega_n)$$

$$\bar{\mu}_\omega = \frac{\langle \omega \rangle_{measured} - \sigma_G^2 \sum_{\omega_a}^{\omega_b} (q_n \omega_n)}{\left| P_m^2 \right| - 2\sigma_G^2 \sum_{\omega_a}^{\omega_b} (q_n)} \pm \sigma_1$$

Conclusion

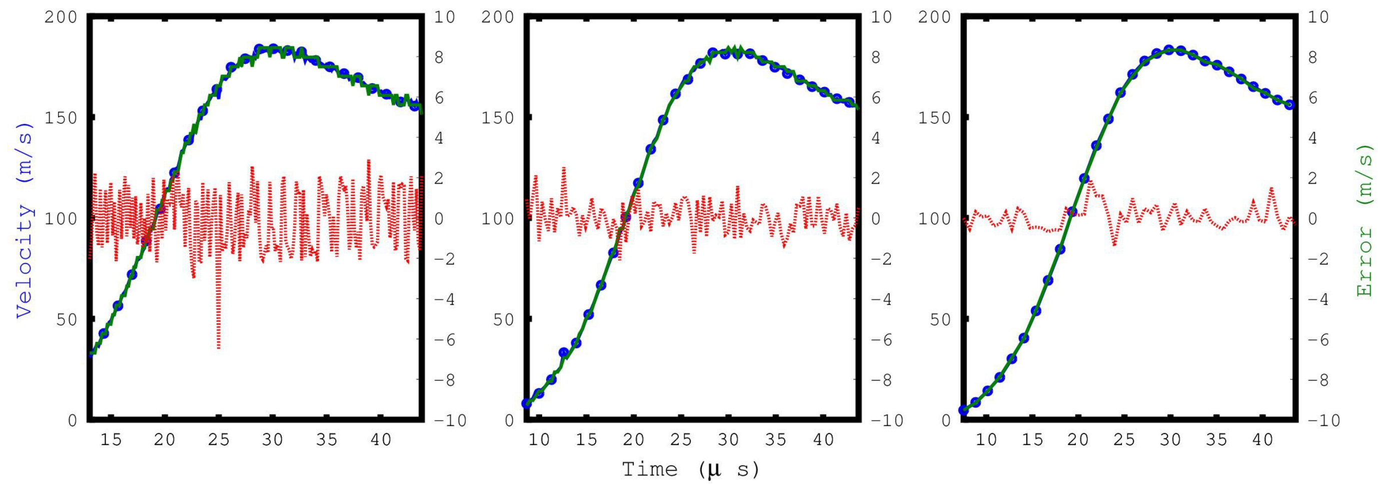
- **Shown analytical treatment using time frequency analysis which demonstrates**
  - Errors are present between the peak, local mean, and true velocity for targets moving with non-linear acceleration
  - The optimal window which minimizes the bandwidth
  - Accuracy error
- **Analyzed the effect of noise in the captured signal**
  - Numerically Gaussian noise → exponential statistic in P^2
  - Use the mean value theorem to bound noise error
  - Shown the utility of thresholding to reduce uncertainty

Real Data



- **Expanding ring at 3 different window sizes**

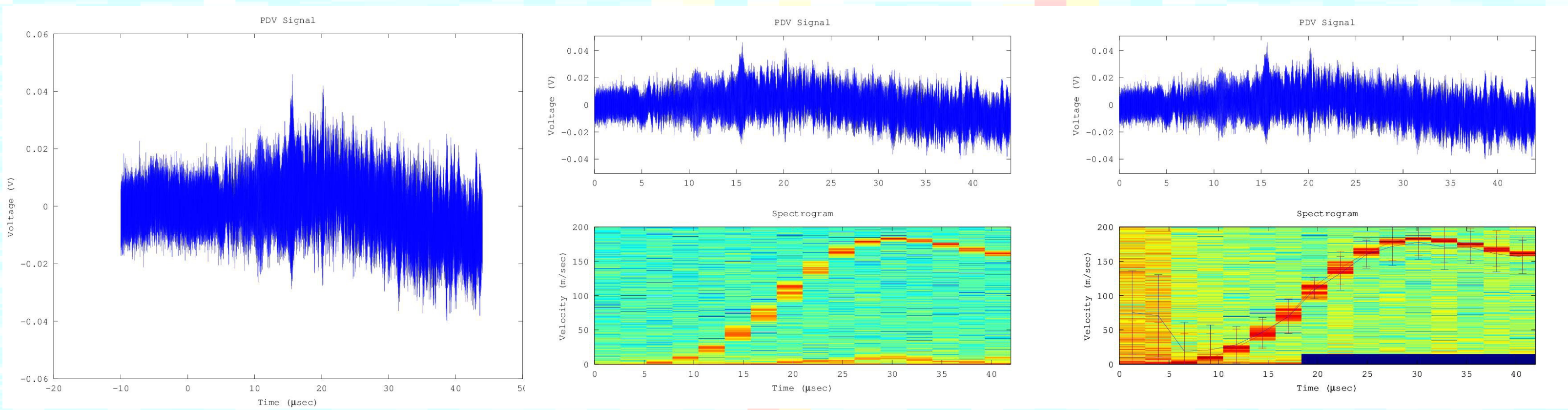
Real Data



- **Difference between mode and local mean frequency**

Steps of Real Analysis

- **Load data**
- **Take spectrogram**
- **Eliminate known bad data**



Steps of Real Analysis

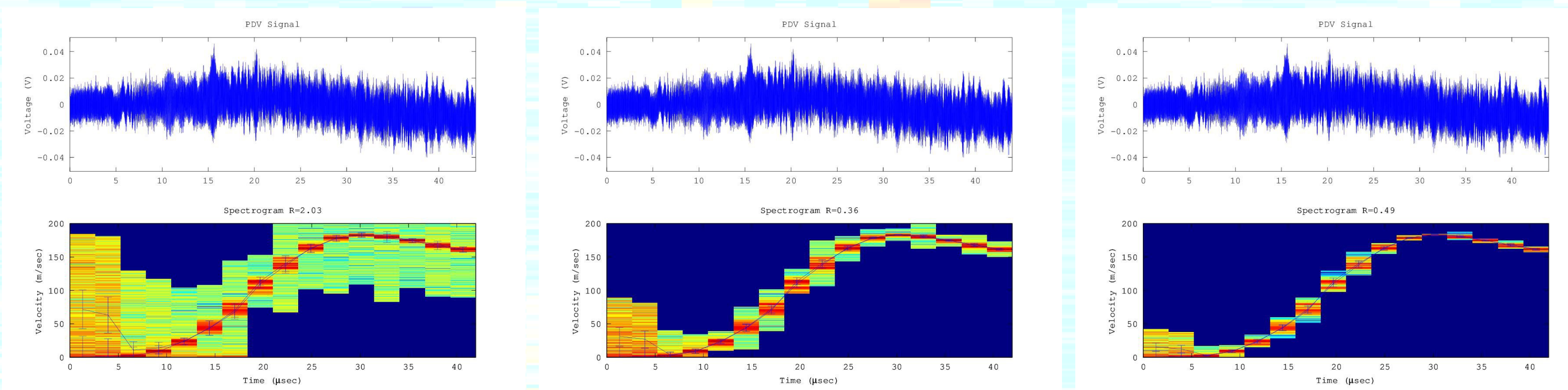
- **Iterate**
  - Calculate mode, μ, σ_v and σ_m while narrowing to 3 σ_v of peak,
  - Iteration stops

    iix = mu2mode < 0.5*sigm;

    ikx = (p > pnoise).';

    (mean(sigm2sigv(iix&ikx)./sigm(iix&ikx))>zcrit);

    count>3 and count < 10



Steps of Real Data

- **Calculate all values**