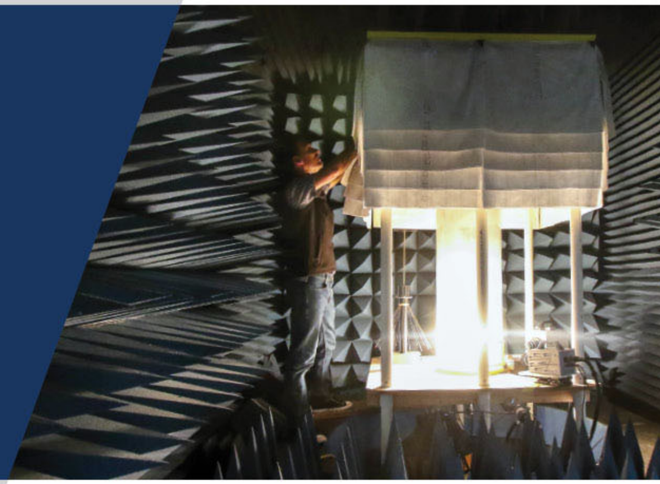
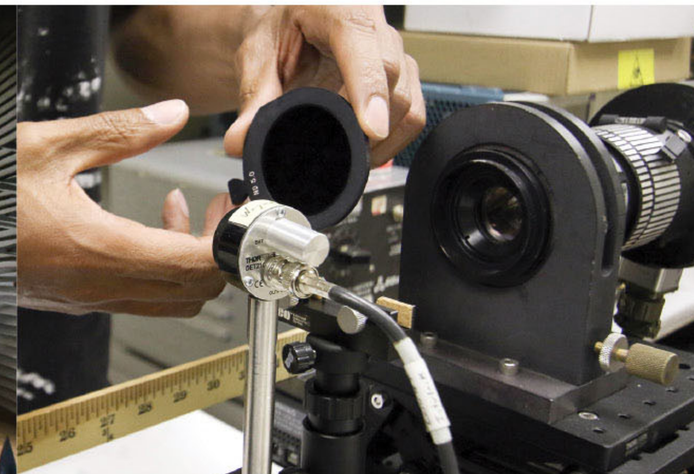
Extending the Asay window measurement model to high areal mass ranges





Continuum mechanics investigation of the Asay window diagnostic

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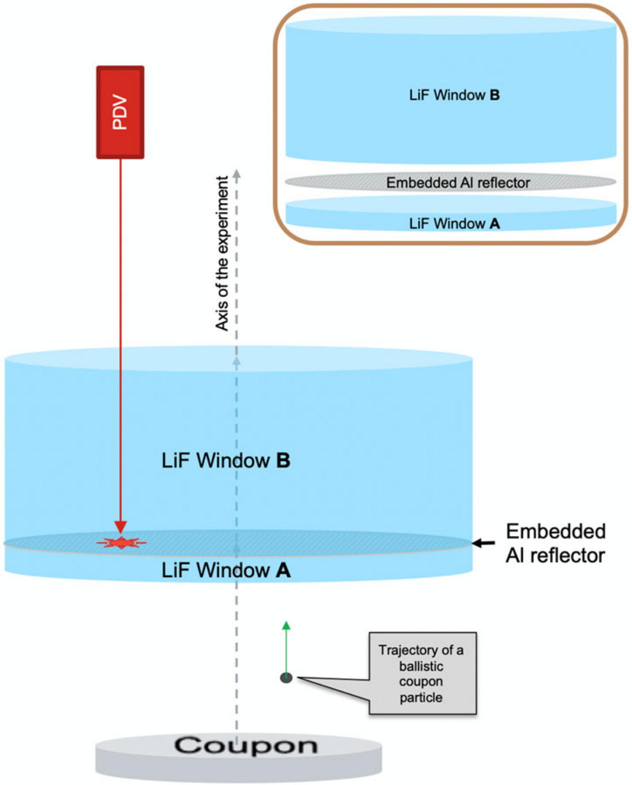
Outline

1. Asay window diagnostic
   - Asay Window areal mass diagnostic
   - Relevant prior work and analysis methods
2. Accretion layer and the variable mass system
3. Accretion Layer modeling options
4. Variable mass system balance of linear momentum
5. Incompressible accretion layer
   - Derivation
   - Measurands
6. Compressible accretion layer
   - Determining accretion layer particle quantities from PDV
   - Derivation and measurands
7. Momentum diagnostic parameter inference
   - Teaser (Thursday session talk)
8. Validation, future work, Conclusions

Asay window apparatus and mass measurement technique

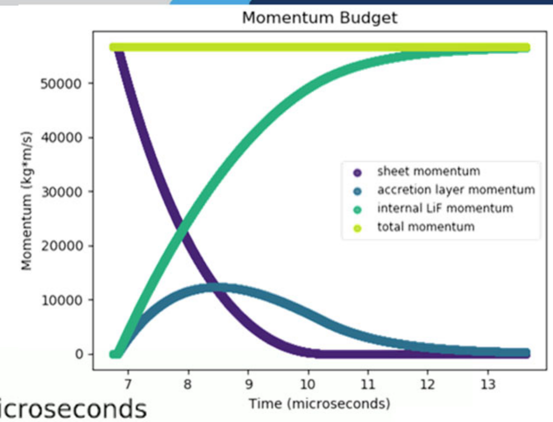
- **Asay window momentum diagnostic:**
  - *Coupon material collides with a (LiF) window, PDV measures velocity at an internal point in the window medium.*
  - Jim Asay introduced the method in 1978: μ
    - Asay, J. R. "Thick-plate technique for measuring ejecta from shocked surfaces."
  - Provided a set of physics assumptions are satisfied, velocimetry analysis can provide ***accumulated mass*** and ***volumetric density measurements***:
    - **Non-interacting, ballistic, and 1D motion of colliding material, simultaneous onset of motion**
    - **Collisions with window are spatially uniform and inelastic,**
    - **Conservation of momentum in the coupon particle + Asay window system**
    - *Colliding material forms a dynamic accretion layer of approx. uniform density*
    - *Collisions occur at a known/consistent location relative to the accretion layer*
- **In the last ~decade this diagnostic has been enhanced through improvements in the characterization of LiF, dual-layer/embedded-reflector windows, and analysis methods:**
  - LiF:
    - (Jensen et al., 2007), (Rigg et al., 2014),…
  - Dual layer window:
    - (Chen, et al. 2017), (Stevens, et al. 2021)
  - Analysis (Spall, porosity, stress):
    - (Chen, et al. 2017), (Chen, et al. 2012), (Stevens, et al. 2021)

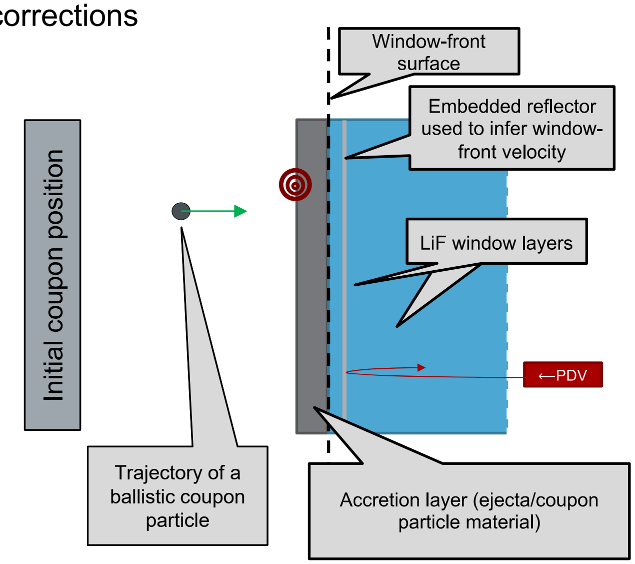


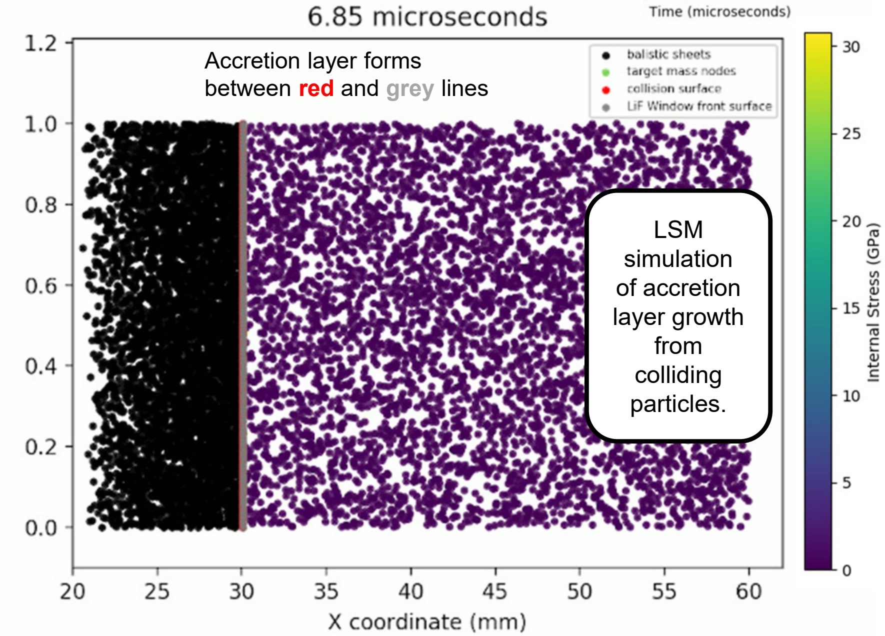
Current work focuses on extending the Asay window areal mass measurements into the high areal mass range (on the order of 1 g/cm^2)

Introducing a dynamic accretion layer to the analysis

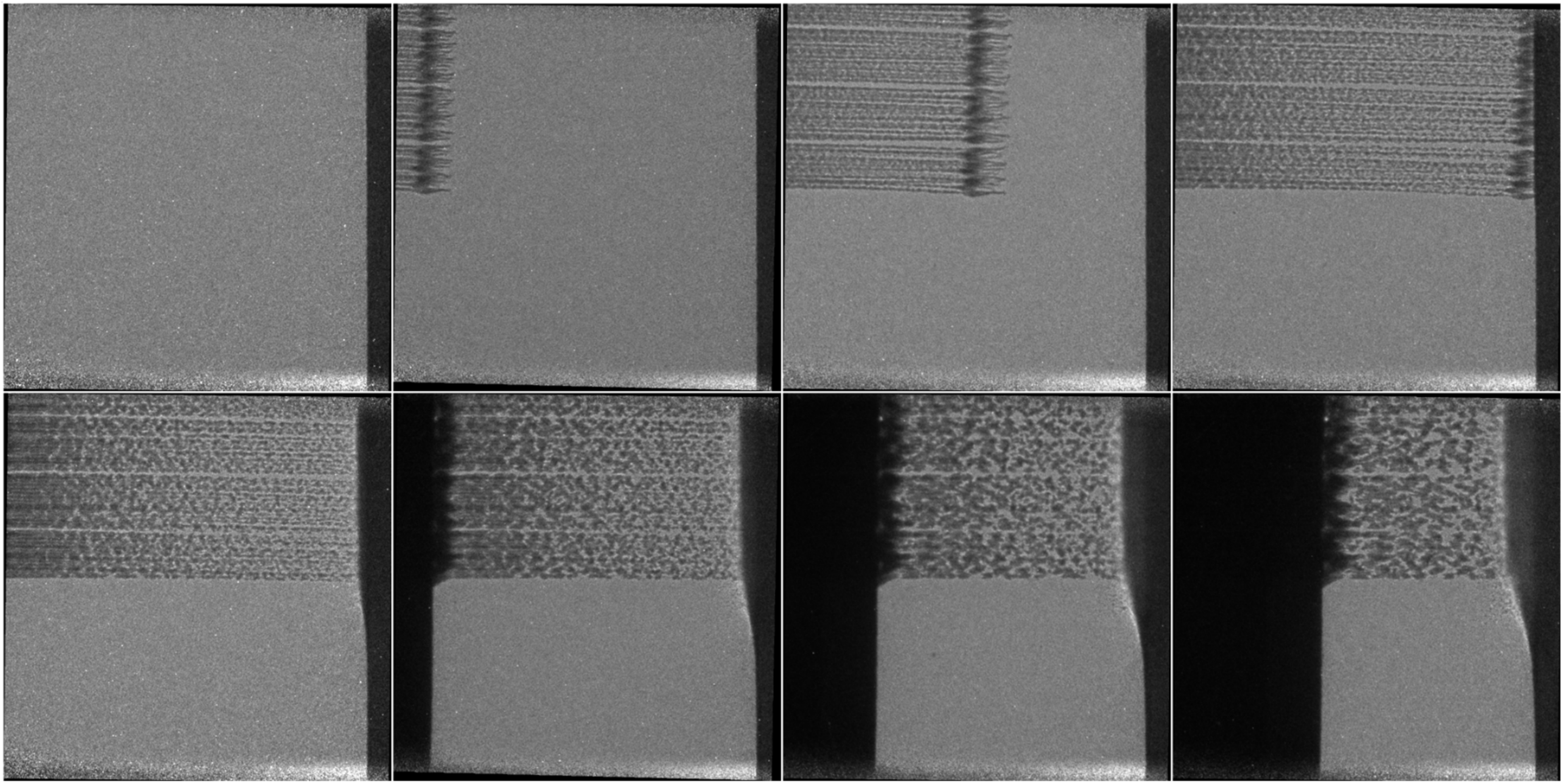
- High quantities of accumulated surface deposits on the window motivate the inclusion of a dynamic layer in the analysis that contains the accumulated mass: *the accretion layer.*
- Assuming accretion layer consists of coupon material, the thickness of the accretion layer could be comparable to the thickness of the pre-motion coupon (mm's).
- If present, the accretion layer (or thickness thereof) influences:
  - Determination of ballistic particle velocity at collision (v_e)
  - Equation of motion and momentum calculations
  - Time corrections



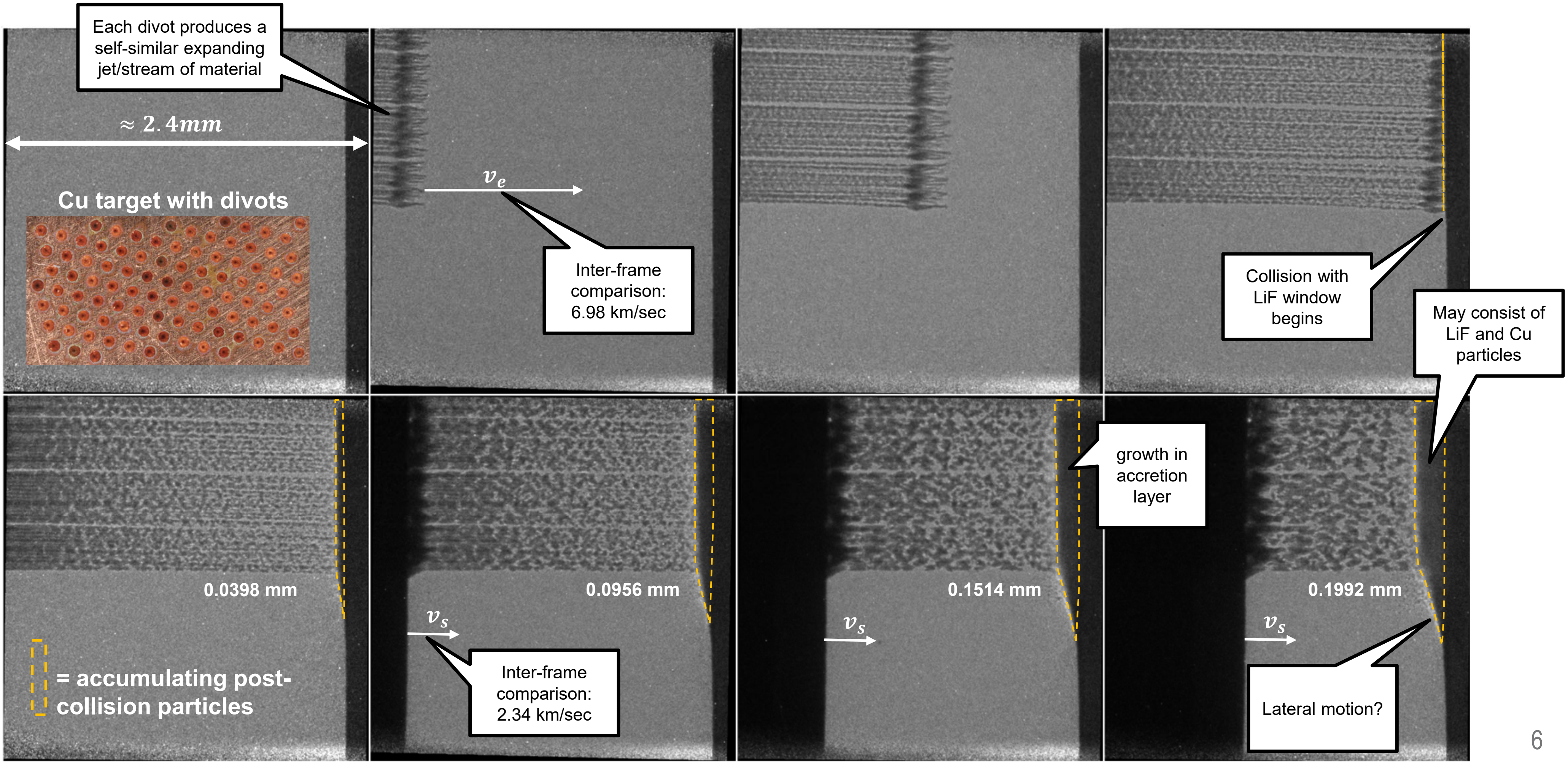




Accretion layer growing on a LiF window: DCS 22-4-023

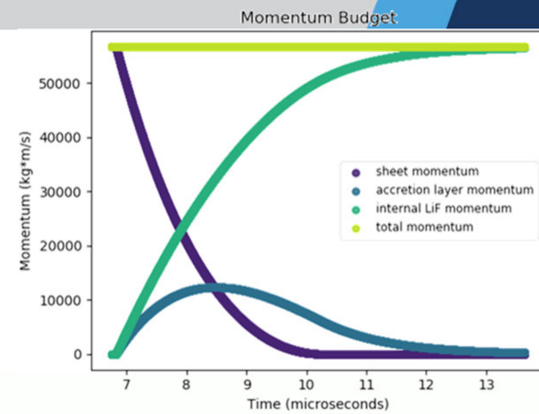


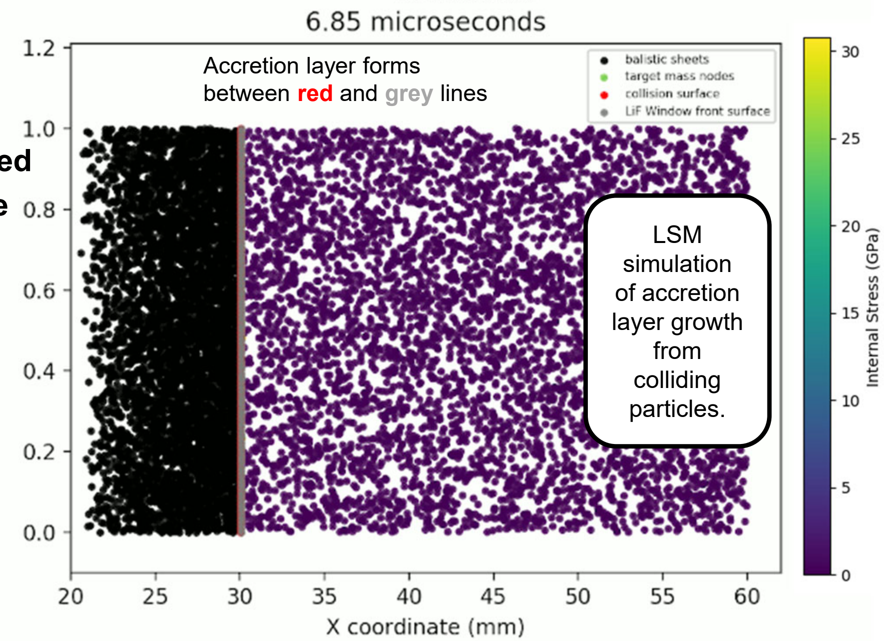
Accretion layer growing on a LiF window: DCS 22-4-023



Accretion Layer Modelling Options

- **No Accretion Layer, the "Null Accretion Model"**
  - **Not unreasonable early in an experiment during low-density accumulation**
- **Dynamic incompressible accretion layer**
  - **Models the accretion layer like a growing-thickness Asay foil**
  - **Areal mass and volumetric density measurements are feasible for some experiments and time ranges (early-mid).**
- **Compressible accretion layer with constant shock propagation velocity**
  - **Can use the bulk sound speed or a mid-experiment shock speed**
  - **Significant improvements in temporal accuracy, especially late in an experiment.**
- **Compressible accretion layer with linear shock-particle velocity relation**
  - **Accretion layer with linear Hugoniot EOS model**
  - **Impedance mismatch between accretion layer and LiF is accounted for (inc. reflections), complex shock propagation in the accretion layer.**





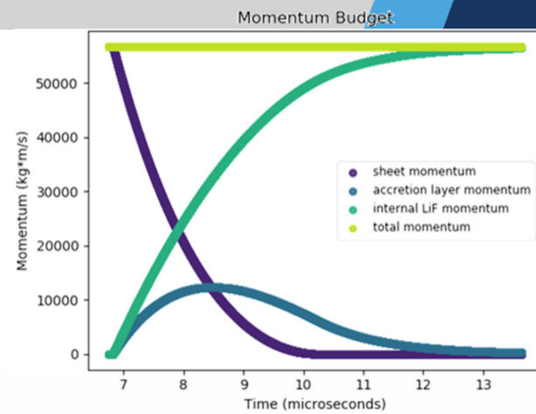
Accretion Layer Modelling Options

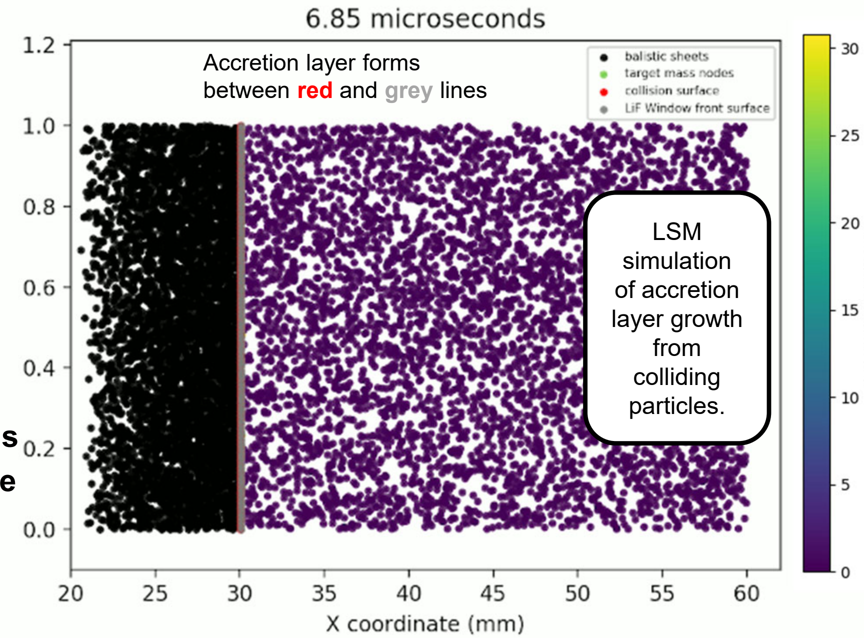
- **No Accretion Layer, the "Null Accretion Model"**
  - **Analysis method: 0**

- **Dynamic incompressible accretion layer**
  - **Analysis methods: numerical integration of an equation of motion (continuum mechanics)**
  - **Low/mid complexity (comp foil analysis)**

- **Compressible accretion layer with constant shock propagation velocity**
  - **Analysis methods: numerical integration of an equation of motion (continuum mechanics)**
  - **Mid-complexity: stronger assumptions on the behavior within the accretion layer.**

- **Compressible accretion layer with linear shock-particle velocity relation**
  - **Analysis methods: forward simulations and inference methods**
  - **High-complexity: strongest assumptions on the behavior of the accretion layer. Positivity of accretion layer mass-flux. "BIE" but for momentum diagnostics.**





Equations of Motion

Approach: balance of linear momentum for variable mass system
Incompressible accretion layer

- **General formulation of the balance of linear momentum for variable mass systems with mass flux, internal flow, external and surface forces (Levi Civita, 1928), (Irschik et al., 2004):**

$$\boldsymbol{M}\cdot\ddot{\boldsymbol{c}} = \boldsymbol{F}_{\boldsymbol{body}} + \boldsymbol{F}_{\boldsymbol{surf}} + \dot{\boldsymbol{M}}\cdot\left(\boldsymbol{v}_{\boldsymbol{e}} - 2\cdot\dot{\boldsymbol{c}} + \frac{\boldsymbol{d}}{\boldsymbol{dt}}\hat{\boldsymbol{c}}\right) - \ddot{\boldsymbol{M}}\cdot(\boldsymbol{c}-\hat{\boldsymbol{c}}) \quad \textbf{[A]}$$

  - $\boldsymbol{M}$: mass of the system,
  - $\boldsymbol{c}$: ***center of mass*** of the system
  - $\hat{\boldsymbol{c}}$: ***center of mass influx*** of the system (where the particles are colliding)
  - $\boldsymbol{v}_{\boldsymbol{e}}$: summarized velocity of the colliding particles.
  - $\boldsymbol{F}_{\boldsymbol{body}}$: external body forces
  - $\boldsymbol{F}_{\boldsymbol{surf}}$: surface forces acting on the system

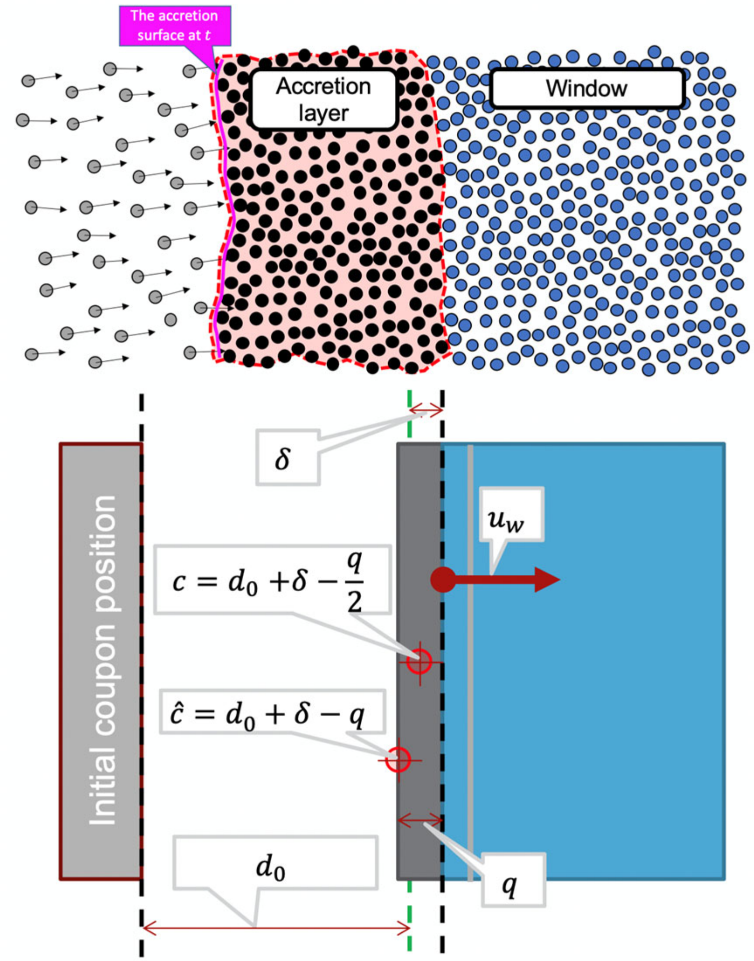
- **We define the control surface describing the variable mass system to be the surface boundary of the accretion layer.**
  - $\boldsymbol{F}_{\boldsymbol{body}}$: Negligible body forces act on the accretion layer (e.g. gravity)
  - $\boldsymbol{F}_{\boldsymbol{surf}}$: Pressure exerted by the window on surface between LiF and accretion layer

- **We additionally assume the following:**
  - ***Uniform density accretion layer approximation***

$$\boldsymbol{c} \approx \boldsymbol{d}_{\boldsymbol{0}} + \int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{u}_{\boldsymbol{w}}(\boldsymbol{\tau})\boldsymbol{d\tau} - \frac{\boldsymbol{q}}{\boldsymbol{2}}$$

  - ***Collisions occur at the accretion surface***

$$\hat{\boldsymbol{c}} \approx \boldsymbol{d}_{\boldsymbol{0}} + \int_{\boldsymbol{0}}^{\boldsymbol{t}} \boldsymbol{u}_{\boldsymbol{w}}(\boldsymbol{\tau})\boldsymbol{d\tau} - \boldsymbol{q}$$



Equation of motion: collisions at accretion layer surface
Incompressible accretion layer

- *The accretion layer thickness is related to areal mass:*

$$q = \frac{M}{\rho_{Sn}} \qquad \frac{dq}{dt} = \frac{\dot{M}}{\rho_{Sn}}$$

- *We can evaluate the $\dot{c}$, $\ddot{c}$, and $\frac{d}{dt}\hat{c}$ using the relationship above*
  - $\dot{c} = u_w - \frac{\dot{M}}{2\rho_{Sn}}$
  - $\ddot{c} = \dot{u}_w - \frac{\ddot{M}}{2\rho_{Sn}}$
  - $\frac{d}{dt}\hat{c} = u_w - \frac{\dot{M}}{\rho_{Sn}}$
- Note that the steps above were done with areal mass M; coordinate frame choice implies $F_{surf} = -\sigma_{LiF}$ when window motion is in positive direction, where σ_{LiF} can be approximated by the Hugoniot jump condition:

$$\boldsymbol{\sigma_{LiF} = \rho_{LiF} \cdot |u_w| \cdot (c_0 + \lambda \cdot |u_w|)}$$

- Substituting the above into Equation A and simplifying results in a first-order separable ODE for accumulated areal mass in terms of experiment observables:

$$\frac{\boldsymbol{dM}}{\boldsymbol{dt}} = \frac{\boldsymbol{M} \cdot \dot{u}_w + \boldsymbol{\sigma_{LiF}}}{\boldsymbol{v_e - u_w}} \quad {}^{*}$$

- ***Uniform density accretion layer approximation***

$$c \approx d_0 + \int_0^t u_w(\tau)d\tau - \frac{q}{2}$$

- ***Collisions occur at the accretion surface***

$$\hat{c} \approx d_0 + \int_0^t u_w(\tau)d\tau - q$$

Balance of linear momentum for variable mass systems :

$$M \cdot \ddot{c} = \cancel{F_{body}} + F_{surf} + \dot{M} \cdot \left(v_e - 2 \cdot \dot{c} + \frac{d}{dt}\hat{c}\right) - \ddot{M} \cdot (c - \hat{c}) \qquad \textbf{[A]}$$

Body forces are negligible

We estimate w using the similar methods as for Asay foils

2nd order mass term cancels with a comparable term produce by $\ddot{c}$

* A comparable continuum mechanics equation of motion for Asay foils can be found in (Tregillis and Harrison, 2020)

Volumetric density

- During a time interval $[t, t + \Delta t)$ a volume V_{flux} of particles collides with the accretion layer. The volumetric density of accreting matter is given by:

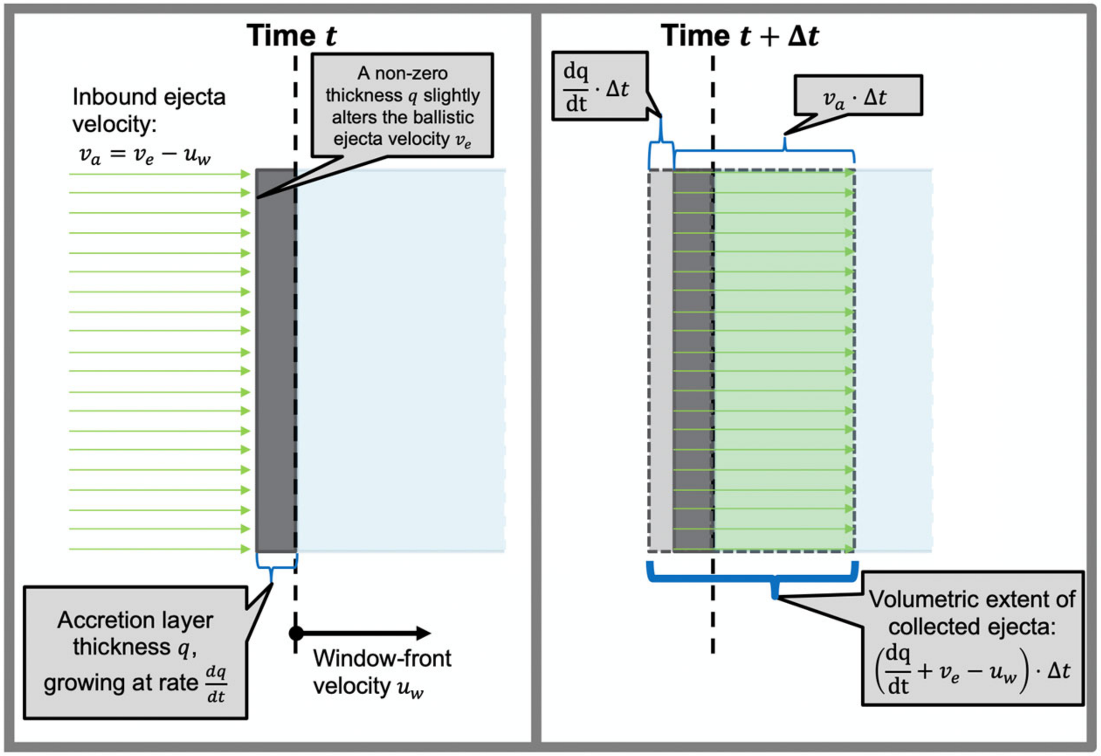
$$\rho_{vol} = \frac{\frac{dM}{dt} \cdot \Delta t}{V_{flux}}$$

- V_{flux} can be expressed in terms of experiment observables and current time-step analysis quantities as:

$$V_{flux} = \left(\frac{dq}{dt} + v_e - u_w\right) \cdot \Delta t$$

- Volumetric density is then given by:

$$\rho_{vol} = \frac{\frac{dM}{dt}}{\left(\frac{dq}{dt} + v_e - u_w\right)} = \frac{M \cdot \dot{u}_w + \sigma_{LiF}}{(v_e - u_w) \cdot \left(\frac{dq}{dt} + v_e - u_w\right)}$$



Extending continuum mechanics eq. of motion to non-rigid (compressible) accretion layer (1D)

- Recall the Asay window equation of motion:

$$\frac{dM}{dt} = \frac{M \cdot \dot{u}_w + \sigma_{LiF}}{v_e - u_w}$$

- Without dynamic time delays the above equation approximates the accretion layer as a rigid body. Introduction of dynamic time delays into the analysis can account for some non-rigid propagation behavior.
- This time-delay approach deviates from strict balance of linear momentum and can be improved upon.
- Will will make use of an approximation for the particle motion withinin the accretion layer to develop an equation of motion for a non-rigid accretion layer.
  - P = set of accretion layer particles at time t
  - For $p \in P$ let m_p = particle mass, x_p = particle position
- Define the ***particle centroid*** and ***particle centroid velocity/accel***:

$$\tilde{c}(t) = \frac{\sum_{p \in P} m_p \cdot x_p}{\sum_{p \in P} m_p} \qquad \frac{d\tilde{c}}{dt}(t) = \frac{\sum_{p \in P} m_p \cdot \dot{x}_p}{\sum_{p \in P} m_p} \qquad \frac{d^2\tilde{c}}{dt^2}(t) = \frac{\sum_{p \in P} m_p \cdot \ddot{x}_p}{\sum_{p \in P} m_p}$$

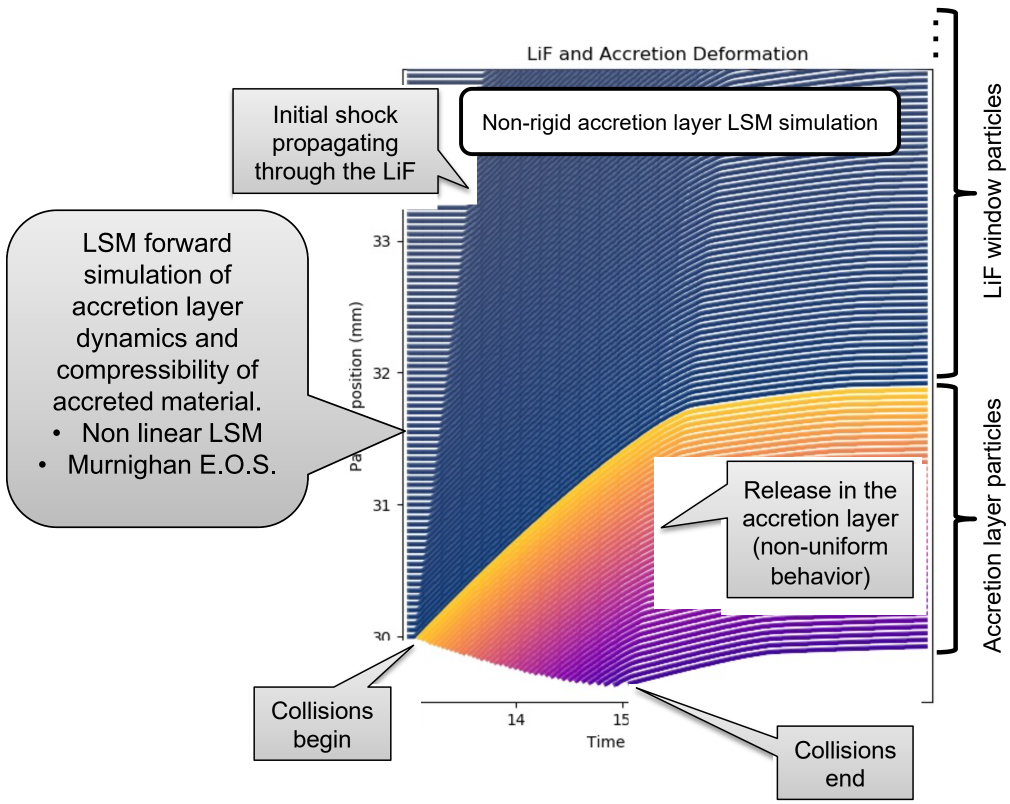
- During mass accretion and for variable mass systems in general: $\frac{d}{dt}\tilde{c} \neq \dot{c}$ (dynamic vs. particle centroid). The following relationship holds relating the $c, \tilde{c}, \hat{c}$ terms:

$$\dot{c} = \frac{d}{dt}\tilde{c} + \frac{\dot{M}}{M} \cdot (\hat{c} - \tilde{c})$$

- Differentiating further and substituting into the balance of linear momentum equation results in the following equation of motion now written in terms of $\tilde{c}$

$$\frac{dM}{dt} = \frac{M \cdot \frac{d^2\tilde{c}}{dt^2} - F_{LiF}}{v_e - \frac{d\tilde{c}}{dt}}$$

- The surface force term is calculated as before ($F_{LiF} = -\rho_{LiF} \cdot u_w \cdot U_S$)



Extending continuum mechanics eq. of motion to non-rigid (compressible) accretion layer (1D)

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- Options for estimating $\tilde{c}, \frac{d\tilde{c}}{dt}, \frac{d^2\tilde{c}}{dt^2}$ from experiment data:
  - **Strategy:** For time t, make use of the observed PDV data for times ***greater than t*** to infer particle velocity at accretion layer internal particles.
  - Use a constant shock propagation velocity U_S^{acc} for the coupon material
    - Let $\boldsymbol{r_L}$ be the Lagrangian coordinate of an accretion layer particle ***p*** ($r_L = 0 = window\ front$)
    - We will observe the velocity of particle **p** in the PDV velocity data at approximately time $t + \boldsymbol{r_L}/U_S^{acc}$, similarly for particle accel.
    - Integrate 1D shock density expressions to estimate Eulerian accretion layer thickness.
    - Options for U_S^{acc} include:
      - $C_0 =$ bulk sound speed
      - $U_S^{acc} = C_0 + \lambda u_p{}'$ for a mid-late observed particle velocity at onset of release/cessation of mass accumulation.
  - Alternative: use forward simulation estimates

Let the antiderivative of window velocity be given by:

$$\delta_w(t) = \int_0^t u_w(s)ds$$

The quantity $\frac{d\tilde{c}}{dt}(t)$ can be written in terms of observable quantities and the assumed accretion layer shock velocity

$$\frac{d\tilde{c}}{dt}(t) \approx \frac{U_S}{Q_L(t)} \int_t^{t+\frac{Q_L}{U_S}} u_w(\tau)d\tau = \frac{U_S}{Q_L} \cdot \left[\delta_w\left(t + \frac{Q_L}{U_S}\right) - \delta_w(t)\right]$$

Similarly, $\frac{d^2\tilde{c}}{dt^2}(t)$ can be evaluated as follows:

$$\frac{d^2\tilde{c}}{dt^2}(t) \approx \frac{U_S}{Q_L(t)} \int_t^{t+\frac{Q_L}{U_S}} \dot{u}_w(\tau)d\tau = \frac{U_S}{Q_L(t)} \cdot \left[u_w\left(t + \frac{Q_L}{U_S}\right) - u_w(t)\right]$$

The growth rate of the Eulerian thickness of the accretion layer is the last tricky quantity:

$$\frac{dQ_E}{dt} = \frac{dQ_L}{dt} - \left[u_w\left(t + \frac{Q_L}{U_S}\right) \cdot \left(1 + \frac{1}{U_S} \cdot \frac{dQ_L}{dt}\right) - u_w(t)\right]$$

Extending continuum mechanics eq. of motion to non-rigid (compressible) accretion layer (1D)… *(Derivation)*

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Balance of linear momentum equation: $M\ddot{c} = F_{surf} + \dot{M}\left(v_e - 2\dot{c} + \frac{d}{dt}\hat{c}\right) - \ddot{M}(c - \hat{c})$ [A]

- $P(t)$ = dynamic collection of accretion layer particles at time t
- For $p \in P$ let m_p = particle mass, x_p = position, $\dot{x}_p$, $\ddot{x}_p$ = velocity, accel.
- $M_0 = M(t) = \sum_{p \in P(t)} m_p$, $\Delta M = M(t + \Delta t) - M(t)$

▶ Define the ***particle centroid*** and ***particle centroid velocity/accel***:

$$\tilde{c}(t) = \frac{\sum_{p\in P} m_p \cdot x_p}{\sum_{p\in P} m_p} \qquad \frac{d\tilde{c}}{dt}(t) = \frac{\sum_{p\in P} m_p \cdot \dot{x}_p}{\sum_{p\in P} m_p} \qquad \frac{d^2\tilde{c}}{dt^2}(t) = \frac{\sum_{p\in P} m_p \cdot \ddot{x}_p}{\sum_{p\in P} m_p}$$

▶ Lemma 1:

$$\frac{d}{dt}(M \cdot c) = M \cdot \frac{d\tilde{c}}{dt} + \dot{M} \cdot \hat{c}$$

▶ Differentiating and rearranging results in the following:

[B]
$$\frac{d^2}{dt^2}(M \cdot c) - \frac{d}{dt}\left(\dot{M} \cdot \hat{c}\right) = \dot{M} \cdot \frac{d\tilde{c}}{dt} + M \cdot \frac{d^2\tilde{c}}{dt^2}$$

▶ The *Balance of linear momentum equation [A] can be re-arranged as:*

[C]
$$\frac{d^2}{dt^2}(M \cdot c) - \frac{d}{dt}\left(\dot{M} \cdot \hat{c}\right) = F_{surf} + \dot{M} \cdot v_e$$

▶ Substitute the right hand side of [B] into the equation above eliminating $c, \hat{c}$ terms. Rearranging we get:

$$\dot{M} = \frac{M \cdot \frac{d^2\tilde{c}}{dt^2} - F_{Surf}}{v_e - \frac{d\tilde{c}}{dt}}$$

▶ Equation [C] is now in terms of derivatives of $\tilde{c}(t)$ and other known quantities from experiment data. Surface force term is calculated as before ($F_{Surf} = -\rho_{LiF} \cdot u_w \cdot U_S$) b/c force pertains to particles at the window front surface (not within the accretion layer).

▶ To compute volumetric density we require an estimate for the rate of change of ***Eulerian thickness of the accretion layer***: $\frac{dq_E}{dt}$. If this can be obtained we have:

$$\rho_{Ejecta}(t) = \frac{\dot{M}}{v_e - u_w + \frac{dq_E}{dt}}$$

Validation, Conclusions, and future work

- Validation:
  - Propelled foam experiments (STL): known density and mass accumulation
  - Multi-diagnostic experiments (STL, DCS): radiography comparisons to Asay window measurements.
  - Lattice based inelastic collision simulations
    - Non-linear LSM models
    - Murnaghan and Birch-Murnaghan equation of state
    - (Grady, 2017), (Hockney and Eastwood, 1988)
- Future/current work in Asay window analysis:
  - Asay window behavior under non-uniform conditions
  - 3D balance of linear momentum and Asay window analysis
  - Data assimilation approach to mass recovery (Thursday session talk)

- Conclusions:
  - The accretion layer addition to Asay window analysis results in new equations of motion that depend on the location of collisions and compressibility properties within the layer.
  - Asay window equations of motion can be represented in terms of experiment observables for both rigid and compressible layer assumptions.
  - Balance of linear momentum for variable mass systems provides a framework for developing equations of motion for Asay window diagnostics under a variety of operating assumptions.
  - The growth in the accretion layer thickness during an experiment substantially influences mass and density measurements when high accumulation of areal mass is present.

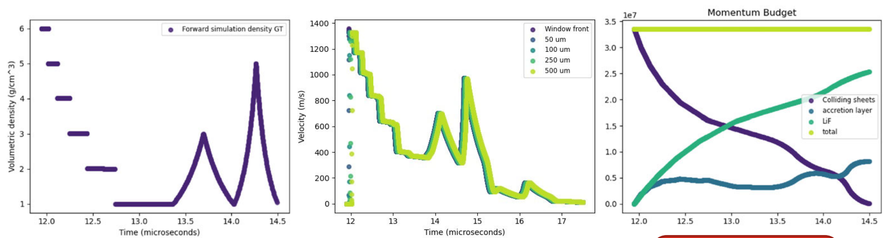
Momentum Diagnostic Parameter Inference

MDPI *– full talk during Thursday session*

- The goal of this work is to determine the ***initial state parameters*** of particles prior to collision with an Asay window that ***best explain*** the observed PDV velocimetry (v_{pdv}) during an experiment.
  - This could be viewed as a *Momentum diagnostic inference engine*
- Continuum mechanics based equations of motion provide one path to parameter inference
  - Integration of the equation of motion to recover ejecta/particle cloud density.
  - Linear elastic accretion layer compressibility
- Forward simulations can provide an alternative path that can relax some physics assumptions used in the continuum mechanics approach
  - Impedance mismatch (accretion layer and Asay window)
  - Linear shock-particle velocity relationship in the accretion layer
  - Positivity
  - **No free lunch**, we must incur/add forward simulation assumptions and accuracy considerations, complexity.
- MDPI formal statement
  - Let P be the set of coupon material particles, for $p \in P$ let $x_p, \dot{x}_p$ be the particle position and velocities.
  - The initial state parameters are $\Theta = [x_p(t=0)]_{p \in P}, [\dot{x}_p(t=0)]_{p \in P}$
  - For a specific instance of initial state parameters, let our forward simulation capability be $F(\Theta) = \mathrm{v}_{\mathrm{simpdv}} = \text{time series}$
  - Inference statement, for an estimate/approximation of the likelihood $\mathcal{L}(\Theta | v_{pdv})$:

$$\Theta_I = \underset{\Theta}{\operatorname{argmax}} \, \mathcal{L}(\Theta | v_{pdv})$$

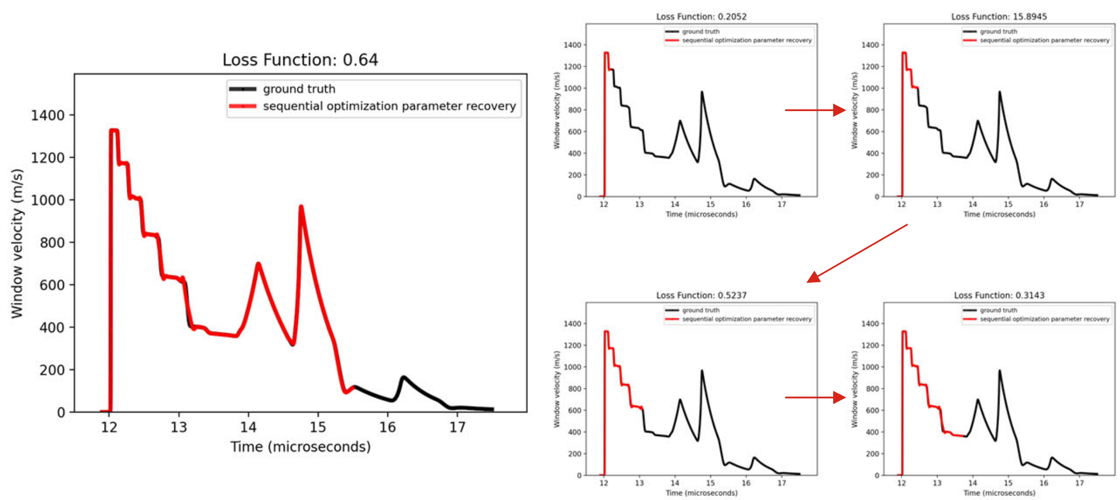
- Current work: data calibration (fit Θ to v_{pdv})
- Can we make use of data assimilation methods in this domain?



Ground truth volumetric cloud density of colliding particles

Simulated PDV data at a selection of embedded reflector points.

Momentum budget



References and Further reading

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Supplementary content

Equation of motion: collisions within the accretion layer
Incompressible accretion layer

- *The accretion layer thickness is related to areal mass:*

$$q = \frac{M}{\rho_{Sn}} \qquad \frac{dq}{dt} = \frac{\dot{M}}{\rho_{Sn}}$$

- *We can evaluate the $\dot{c}$, $\ddot{c}$, and $\frac{d}{dt}\hat{c}$ using the relationship above*
  - $\dot{c} = u_w - \frac{\dot{M}}{2\rho_{Sn}}$
  - $\ddot{c} = \dot{u}_w - \frac{\ddot{M}}{2\rho_{Sn}}$
  - $\frac{d}{dt}\hat{c} = u_w - \frac{\dot{M}}{\rho_{Sn}}$
- Note that the steps above were done with areal mass M; we replace the $-F_{surf}$ term with stress σ_{LiF} and approximate with the Hugoniot jump condition:

$$\boldsymbol{\sigma_{LiF} = \rho_{LiF} \cdot u_w \cdot (c_0 + \lambda \cdot u_w)}$$

- Substituting the above into Equation A and simplifying results in a second-order ODE for accumulated areal mass in terms of experiment observables:

$$\boldsymbol{M \cdot \dot{u}_w = F_{surf} + \dot{M} \cdot (v_e - u_w) + h_0 \cdot \ddot{M}}$$

- ***Uniform density accretion layer approximation***

$$\boldsymbol{c \approx d_0 + \int_0^t u_w(\tau)d\tau - \frac{q}{2}}$$

- ***Collisions at a depth h_0 within the accretion layer***

$$\boldsymbol{\hat{c} \approx d_0 + \int_0^t u_w(\tau)d\tau - (q - h_0)}$$

Balance of linear momentum for variable mass systems :

$$M \cdot \ddot{c} = \cancel{F_{body}} + F_{surf} + \dot{M} \cdot \left(v_e - 2 \cdot \dot{c} + \frac{d}{dt}\hat{c}\right) - \ddot{M} \cdot (c - \hat{c}) \qquad \text{[A]}$$

Body forces are negligible

We estimate w using the similar methods as for Asay foils

2nd order mass term cancels with a comparable term produce by $\ddot{c}$

Non-rigid accretion layer considerations and timing

- A numerical approximation for the particle motion in the Asay window and accretion layer can be used to develop an equation of motion for non-rigid accretion layer.
  - P = set of window and accretion particles at time t,
  - For $p \in P$ let m_p = particle mass.
- Static centroid and centroid velocity:

$$\tilde{c}(t) = \frac{\sum_{p \in P} m_p \cdot p}{\sum_{p \in P} m_p} \qquad \frac{d}{dt}\tilde{c}(t) = \frac{\sum_{p \in P} m_p \cdot \dot{p}}{\sum_{p \in P} m_p}$$

- During mass accretion, $\frac{d}{dt}\tilde{c} \neq \dot{c}$ (dynamic vs. static centroid). The following relationship holds relating the $c, \tilde{c}, \hat{c}$ terms:

$$\dot{c} = \frac{d}{dt}\tilde{c} + \frac{\dot{M}}{M} \cdot (\hat{c} - \tilde{c})$$

- Substituting into the balance of linear momentum equation results in the following equation of motion:

$$M\frac{d^2\tilde{c}}{dt^2} = \dot{M} \cdot \left(v_e - \frac{d\tilde{c}}{dt}\right)$$

- No surface force term because the material volume of the variable mass system is taken to be the entirety of the Asay window and accretion layer.