Using the Complex Spectrum to Analyze PDV Data

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2026 PDV Workshop

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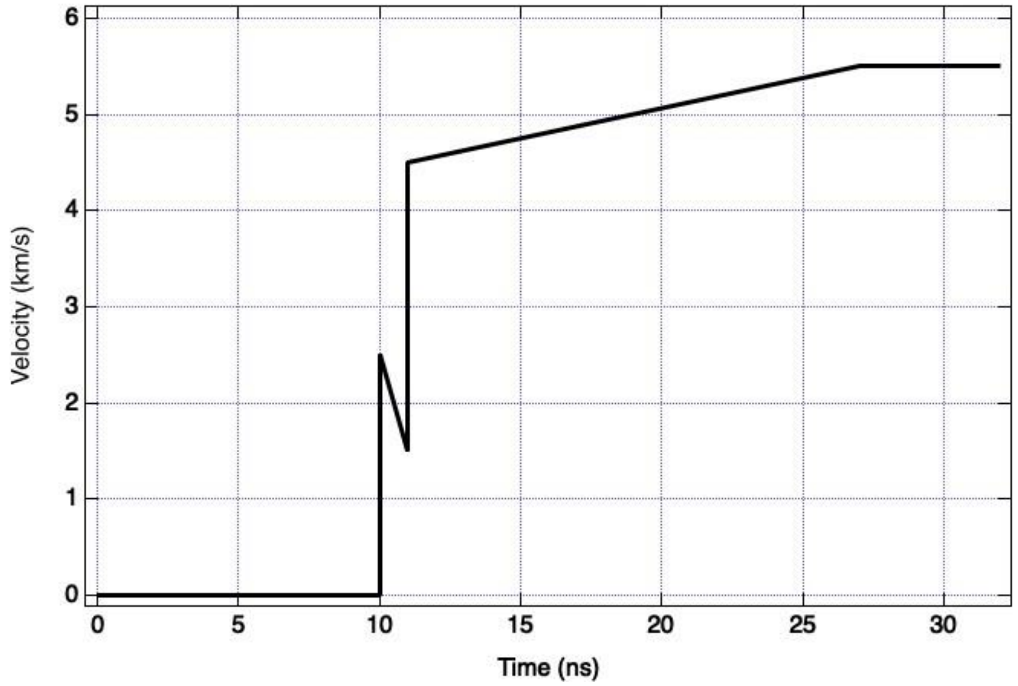
What information is in the complex spectrum?

- **Inspiration**
  - Temporal information is available in the complex part of the Fourier Transform

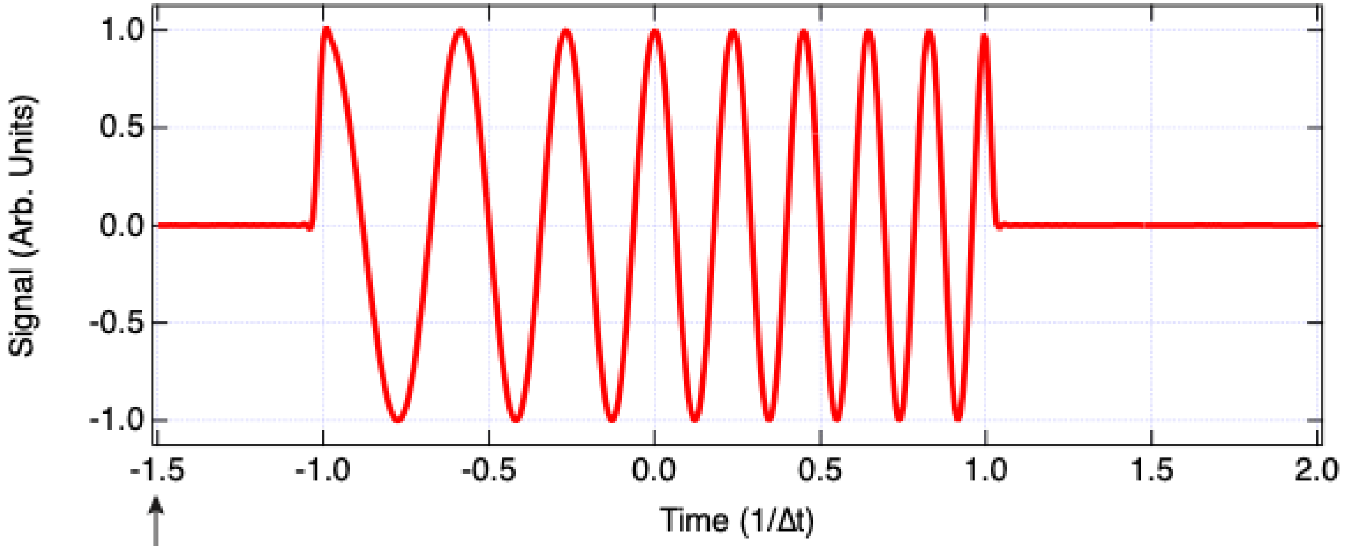
$$\overbrace{f(t) \longleftarrow - - - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt - - - - - \longrightarrow F(\omega)}^{\text{Full temporal/spectral content}}$$

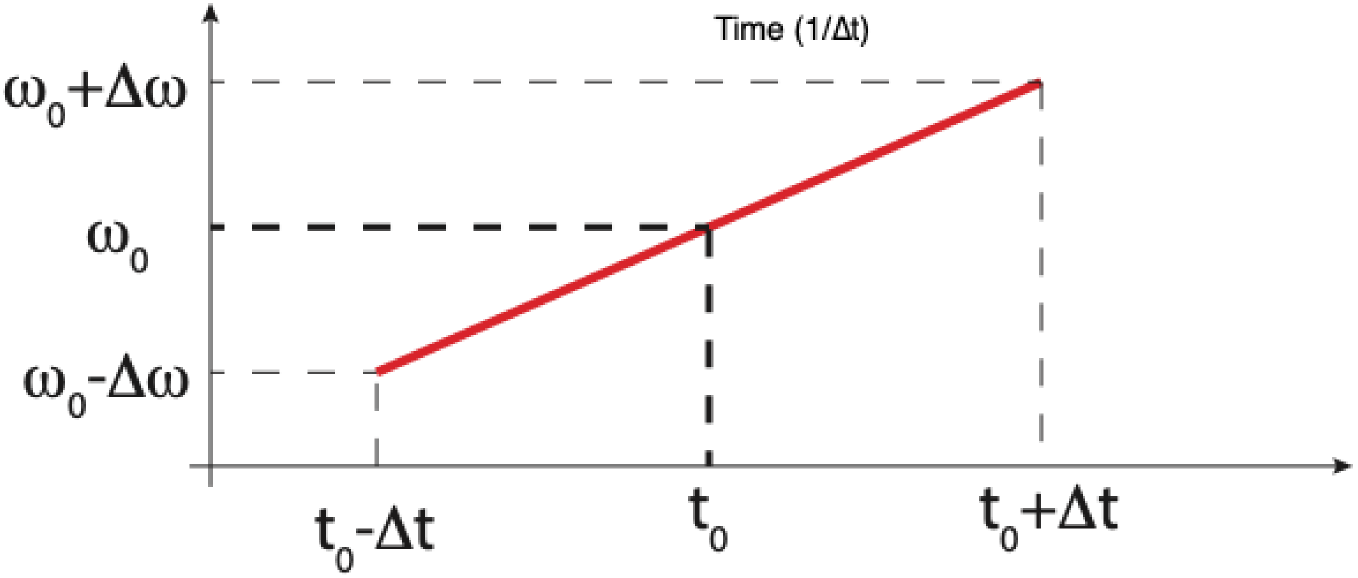
- **Motivation**
  - Resolving discrete changes in frequency in a PDV spectrum
    - Multi-wave structure
    - Pull back



Approach

- **Quanta of Signal**
  - **Amplitude of signal is constant**
  - **Signal has a constant variation in frequency over signal envelop**
  - **Described by 4 quantum numbers:**
    - t_0 - center of envelop
    - Δt- envelop half width
    - ω_0 - central frequency
    - $\Delta\omega$- spectral half width





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$$s(\tau) = \underbrace{\begin{cases} 0 & |\tau| > \Delta t \\ s_0(\tau) & |\tau| < \Delta t \end{cases}}_{\text{Envelop Function}}$$

$$\tau = \underbrace{t - t_0}_{\text{Temporal Offset}}$$

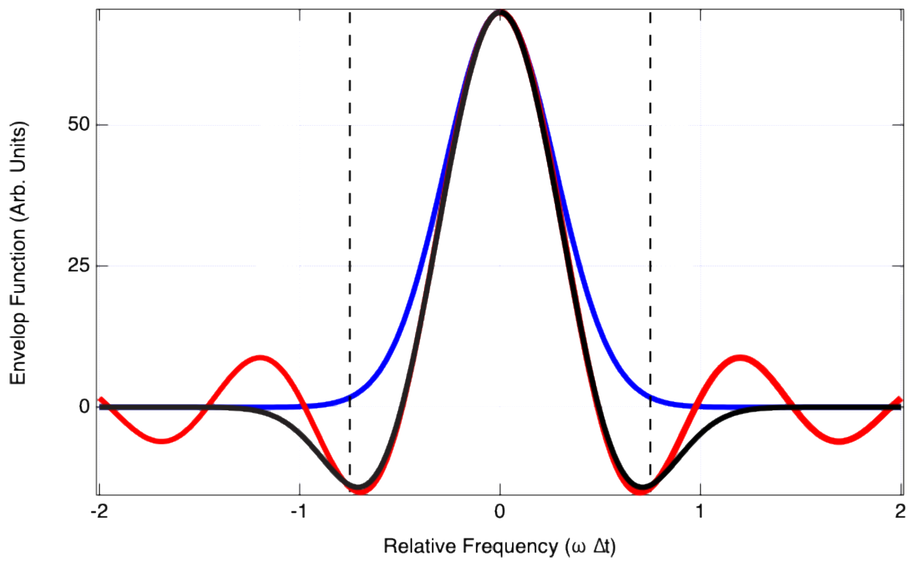
$$\begin{aligned} s_0(\tau) &= A\frac{1}{2}\left(\exp\left[\mathbf{i}\left(\omega_0\tau + \frac{\Delta\omega}{\Delta t}\tau^2 + \phi\right)\right] + \exp\left[-\mathbf{i}\left(\omega_0\tau + \frac{\Delta\omega}{\Delta t}\tau^2 + \phi\right)\right]\right) \\ &= \underbrace{A\frac{1}{2}}_{\text{Amplitude}}\left(\underbrace{\exp\left[-\mathbf{i}\phi\right]}_{\text{Phase}} \cdot \underbrace{\exp\left[-\mathbf{i}\omega_0\tau\right]}_{\text{Oscillations}} \cdot \underbrace{\exp\left[-\mathbf{i}\frac{\Delta\omega}{\Delta t}\tau^2\right]}_{\text{Chirp}} + \underbrace{\exp\left[\mathbf{i}\phi\right]\exp\left[\mathbf{i}\omega_0\tau\right]\exp\left[\mathbf{i}\frac{\Delta\omega}{\Delta t}\tau^2\right]}_{\text{Negative Peak}}\right) \end{aligned}$$

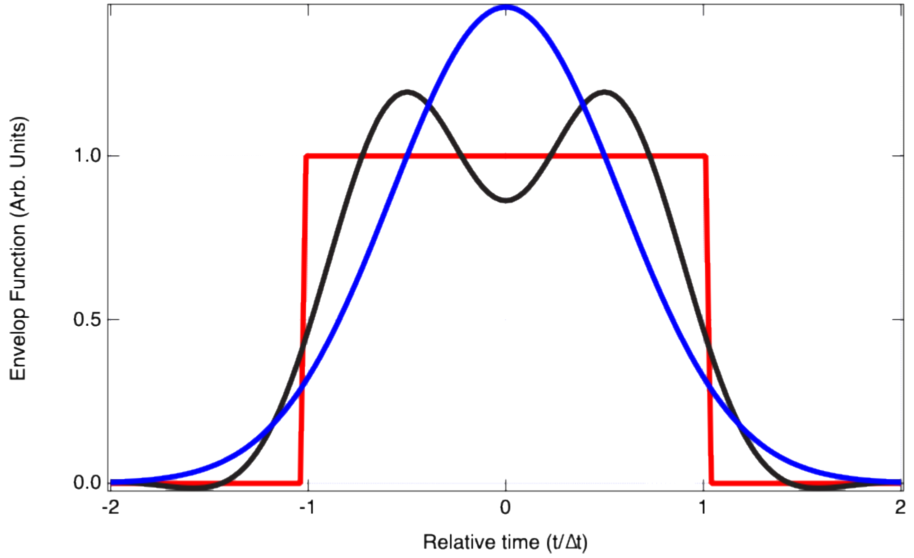
Envelop Function

- **Use a multi-Gaussian fit.**
  - The zero crossings introduce a π-phase shift which is important when working with the complex spectrum

$$\frac{A(\omega)}{A_0^*} = \begin{cases} \operatorname{sinc}(\Delta t \omega) & \text{Direct Fourier Transform of } a(t) \\ \exp\left[-\frac{\Delta t^2 \omega^2}{6}\right] & \text{A Gaussian fit to the sinc function} \\ \exp\left[-\frac{\Delta t^2 \omega^2}{6}\right] - \frac{1}{4}\exp\left[-\frac{\Delta t^2}{4}\left(\omega - \frac{4}{\Delta t}\right)\right] - \frac{1}{4}\exp\left[-\frac{\Delta t^2}{4}\left(\omega + \frac{4}{\Delta t}\right)\right] & \text{Multi-Gaussian fit good to } |\omega|\Delta t < \frac{3}{4} \end{cases}$$

$$A_0^* = \sqrt{\frac{2}{\pi}} A_0 \Delta t$$





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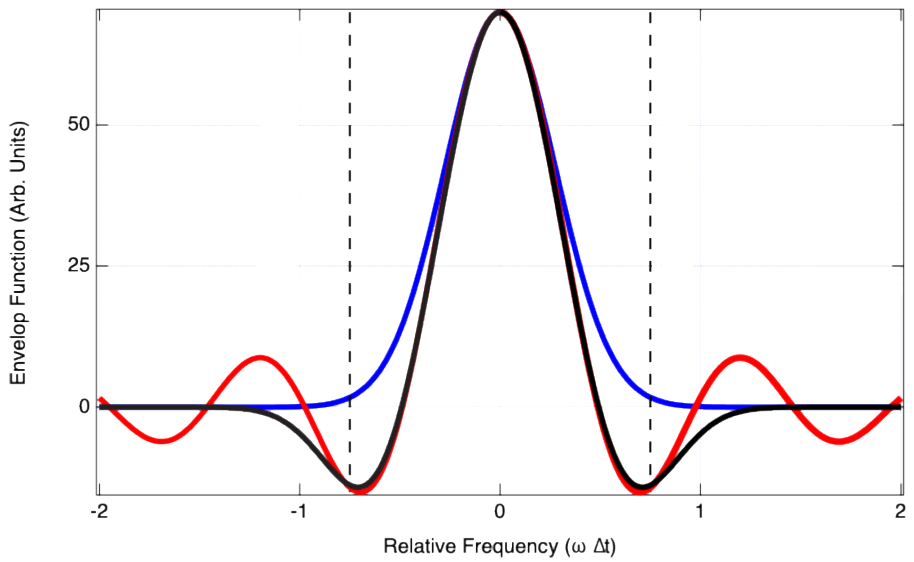
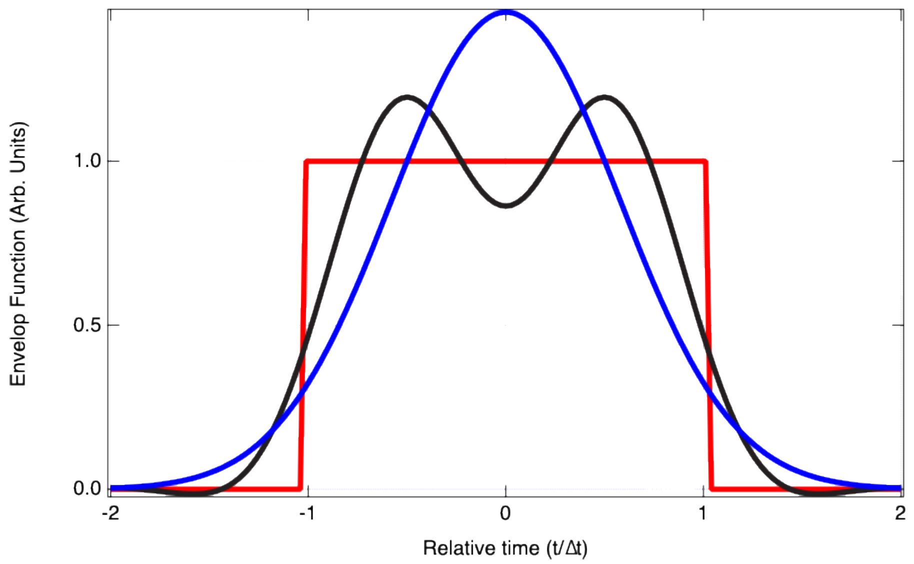
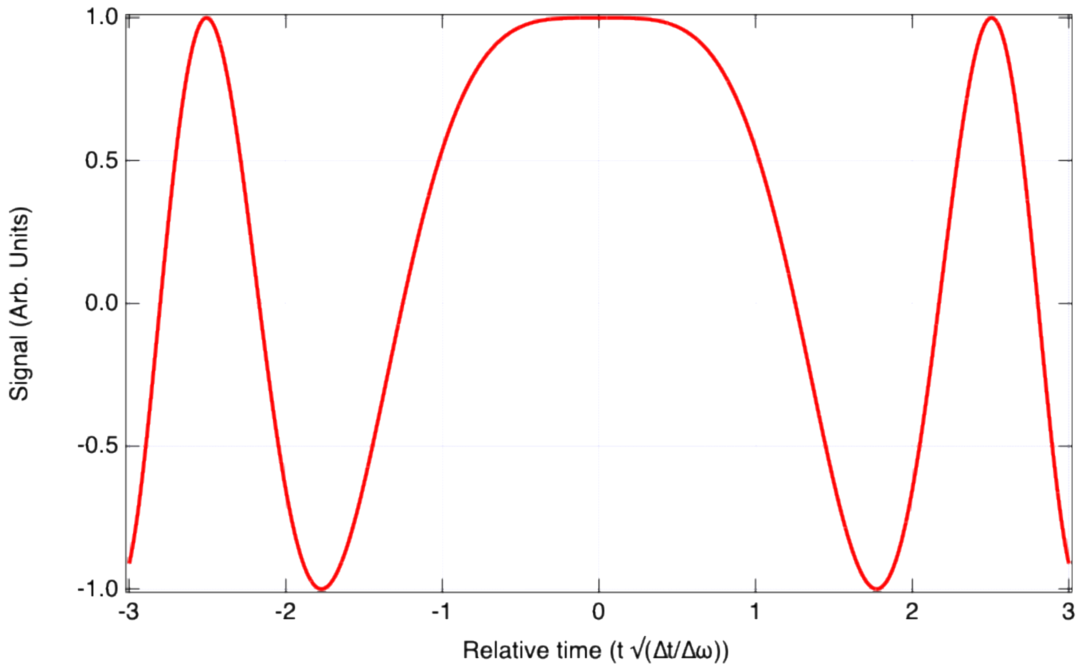
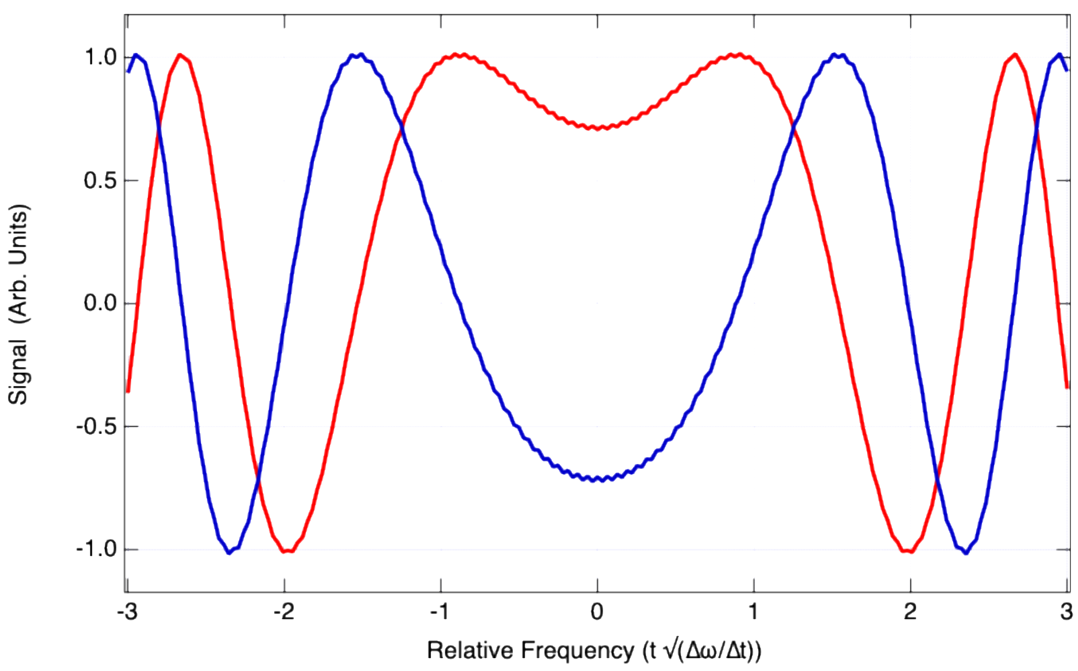




Chirp

- **Direct Fourier Transform of the Chirp component**





$$C(\omega) = \sqrt{\frac{\mathbf{i}\Delta t}{2\Delta\omega}} \exp\left[-\mathbf{i}\frac{\Delta t}{\Delta\omega}\frac{\omega^2}{4}\right]$$

Frequency Representation of Signal Quanta

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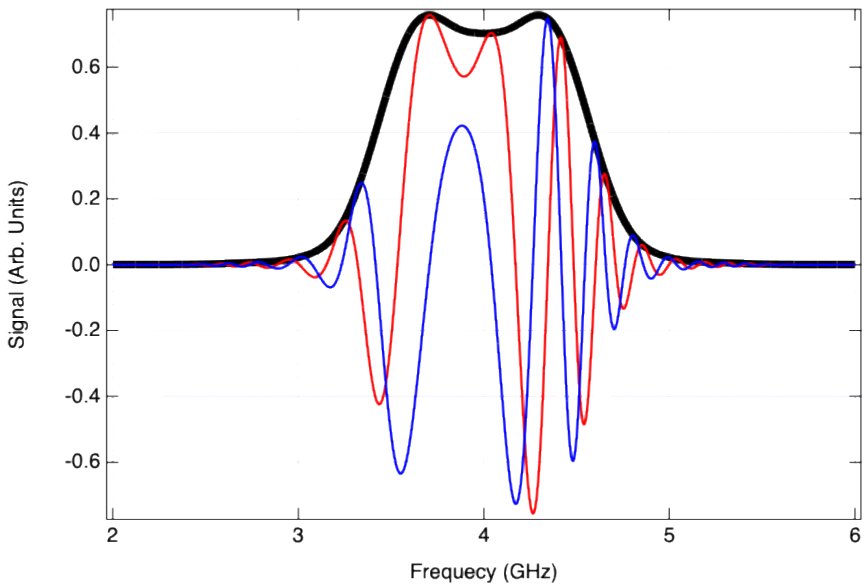
- **Convolution of the fourier transform of the signal components yields:**



$$
\begin{aligned}
S(\omega) &= \Phi_0 \underbrace{\exp[-\mathbf{i}\,\omega\, t_0]}_{\text{Linear Phase Term}} \Bigg\{ \underbrace{\exp\left[-\frac{\Omega^2}{W_0^2(\Delta t,\Delta\omega)}\right]}_{\text{Envelop}} \underbrace{\exp\left[-\mathbf{i}\frac{\Omega^2}{P_0^2(\Delta t,\Delta\omega)}\right]}_{\text{Chirp Phase}} \\
&\quad -\Phi_1 \Bigg(\underbrace{\exp\left[-\frac{(4+\Delta t\Omega)^2}{W_1^2(\Delta t,\Delta\omega)}\right]}_{\text{Envelop}} \underbrace{\exp\left[-\mathbf{i}\frac{(4+\Delta t\Omega)^2}{P_1^2(\Delta t,\Delta\omega)}\right]}_{\text{Chirp Phase}} \\
&\quad + \underbrace{\exp\left[-\frac{(4-\Delta t\Omega)^2}{W_1^2(\Delta t,\Delta\omega)}\right]}_{\text{Envelop}} \underbrace{\exp\left[-\mathbf{i}\frac{(4-\Delta t\Omega)^2}{P_1^2(\Delta t,\Delta\omega)}\right]}_{\text{Chirp Phase}} \Bigg) \Bigg\}
\end{aligned}
$$

$$
\Omega = \omega - \omega_0
$$

Affect of Chirp on Complex Spectrum

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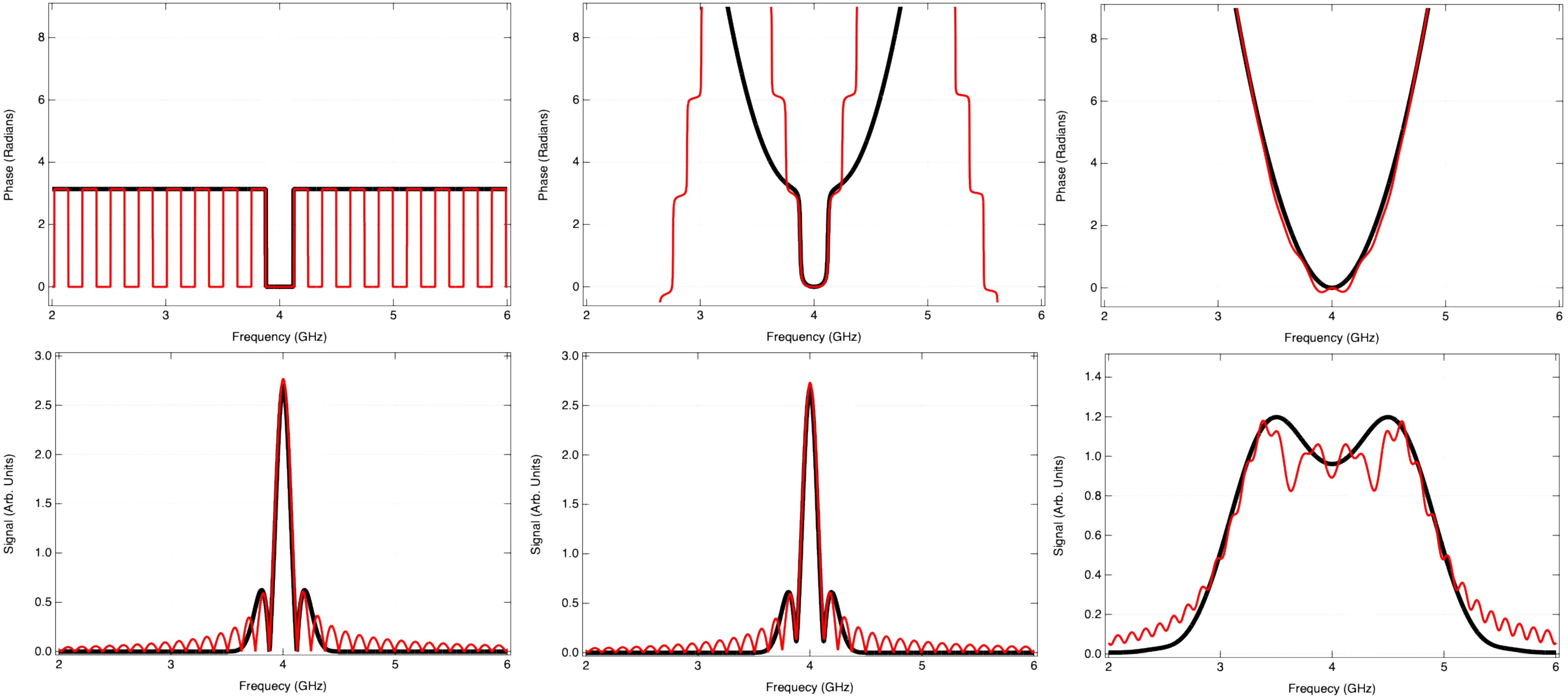
F0 = 4 T0 = 0
ΔT = 4 ΔF = 0.0

F0 = 4 T0 = 0
ΔT = 4 ΔF = 0.01

F0 = 4 T0 = 0
ΔT = 4 ΔF = 0.5



Affect of Chirp on Complex Spectrum

F0 = 4 T0 = 0
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F0 = 4 T0 = 0
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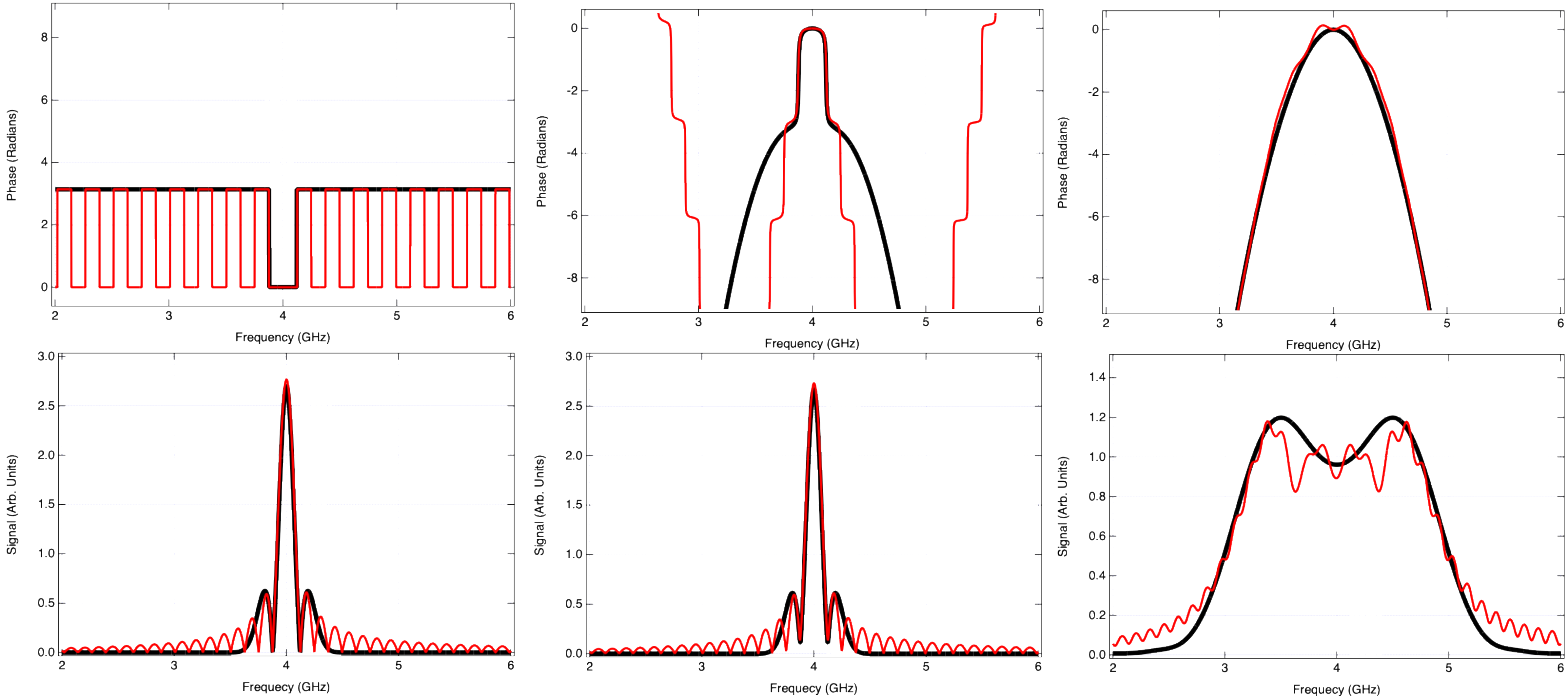
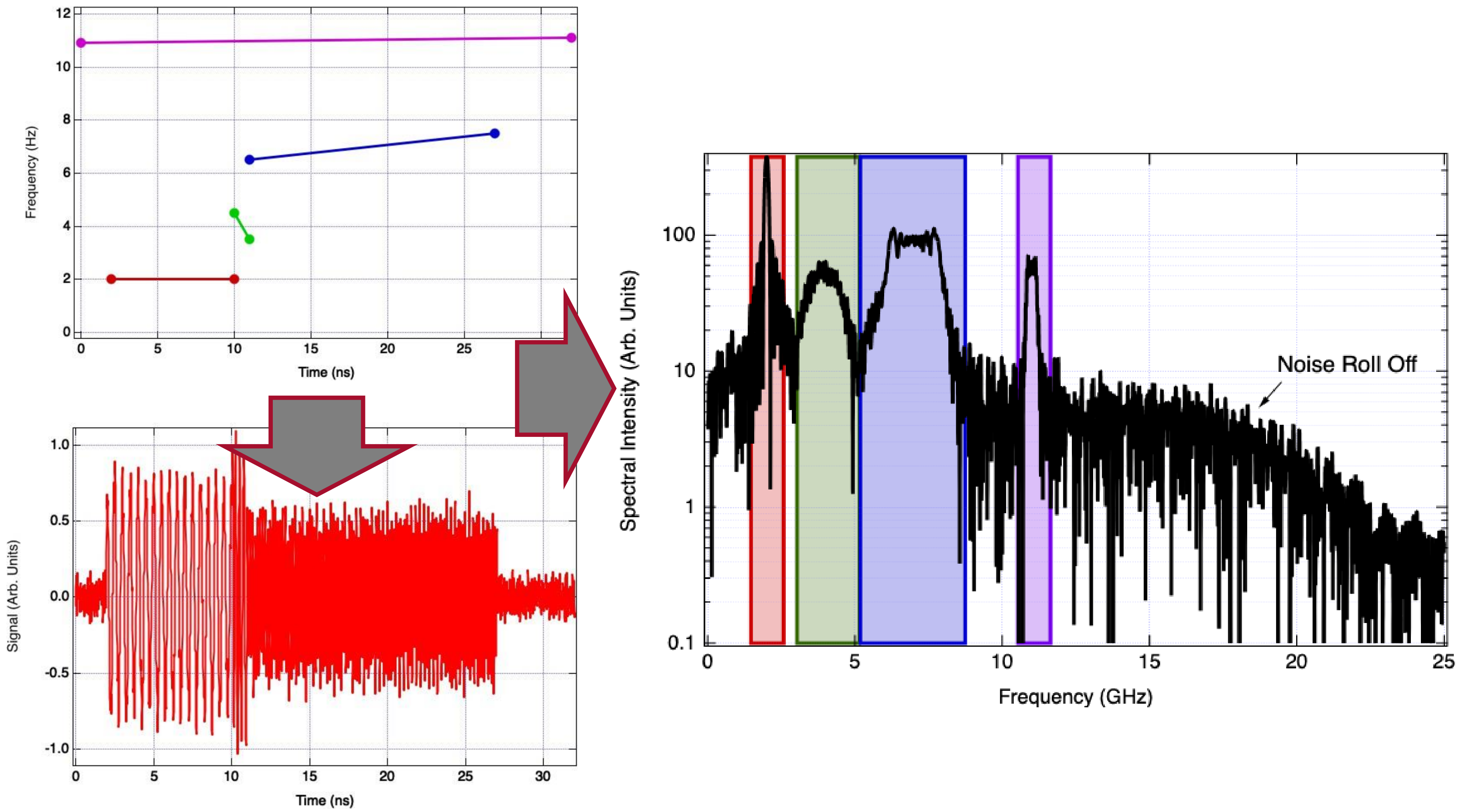


Test Problem

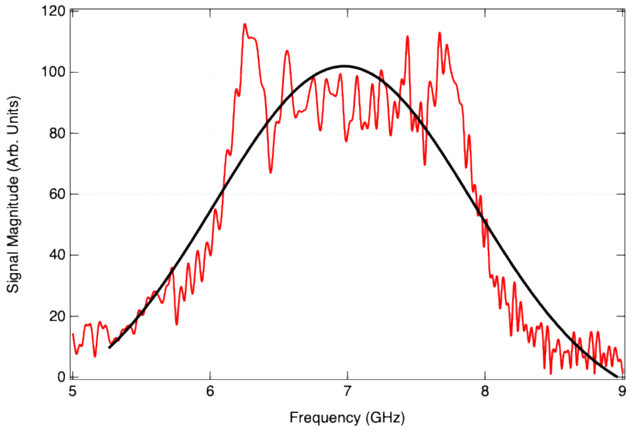
- **Signal Parameters**
  - Discrete Features:
    - Ambient state
    - Initial shock with decay
    - Second wave with gradual increase
    - Additional signal with slow increase
  - Digitizer
    - 128 Gs/sec
    - 4096 points
    - 32 ns window
    - 5% noise fraction



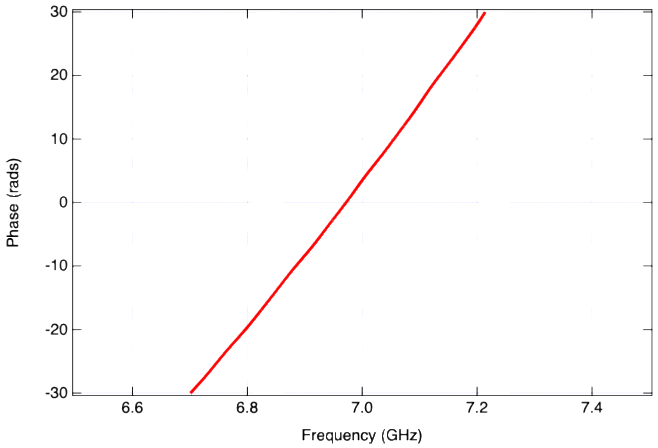
Methodology

- **Fourier Transform of the signal** $S(\omega) = \mathcal{F}[s(t)]$
- **Extract the phase and magnitude** $\varphi(\omega) = \arctan^*(\mathrm{IM}[S(\omega)], \mathrm{Re}[S(\omega)])$

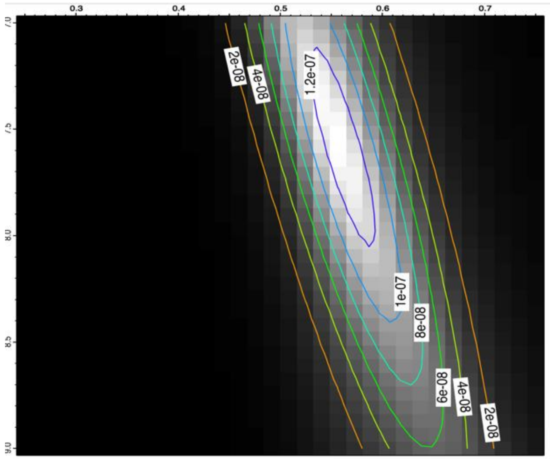
- **Simple Gaussian peak fit to get** f_0 **.**



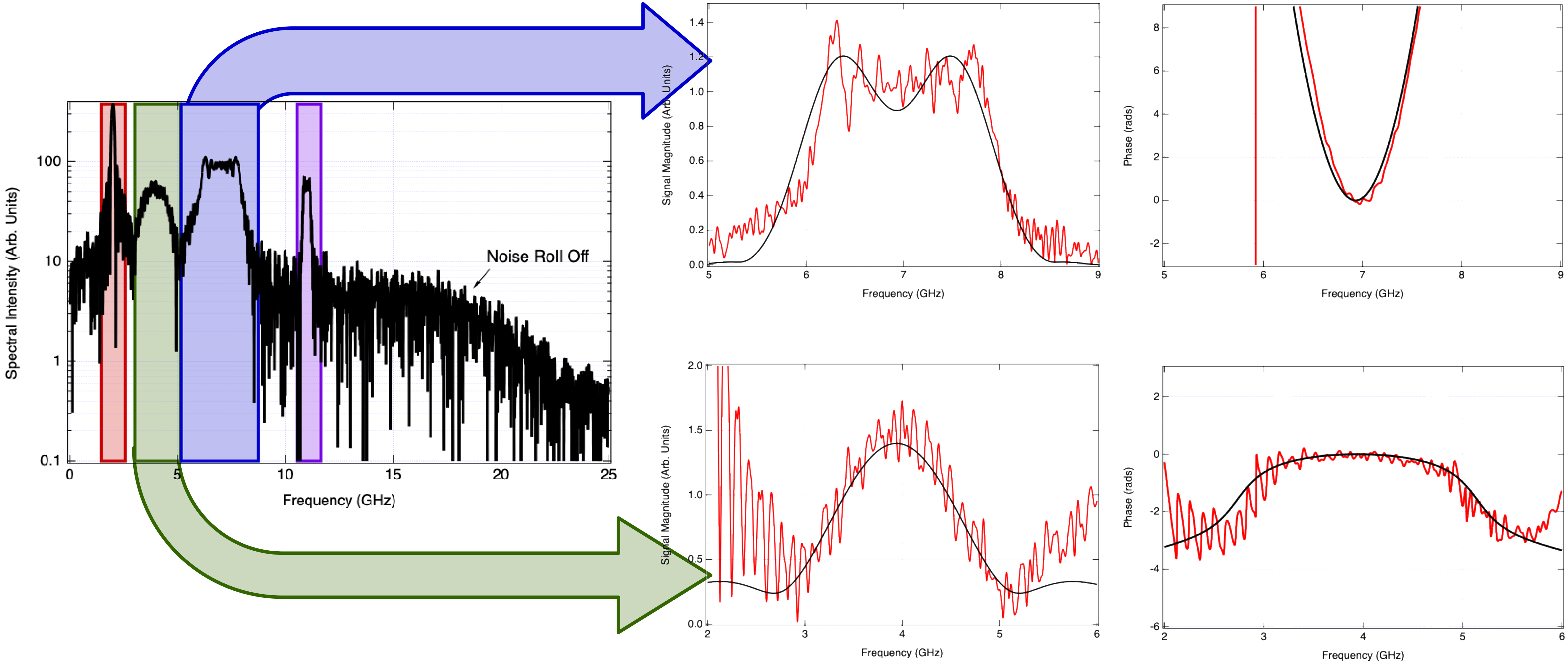
- **Linear fit in frequency of phase gives** t_0 **.**



- χ^2**- minimization to fit** $\Delta t, \Delta\omega$**.**



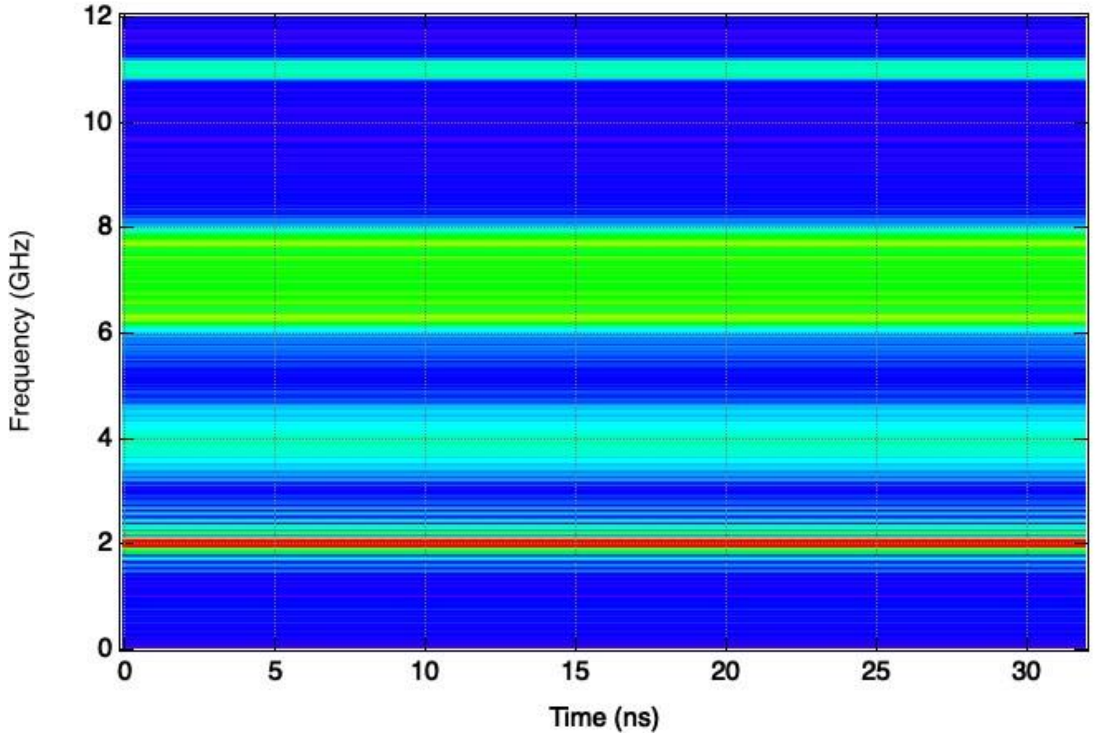
Complex Analysis of signal



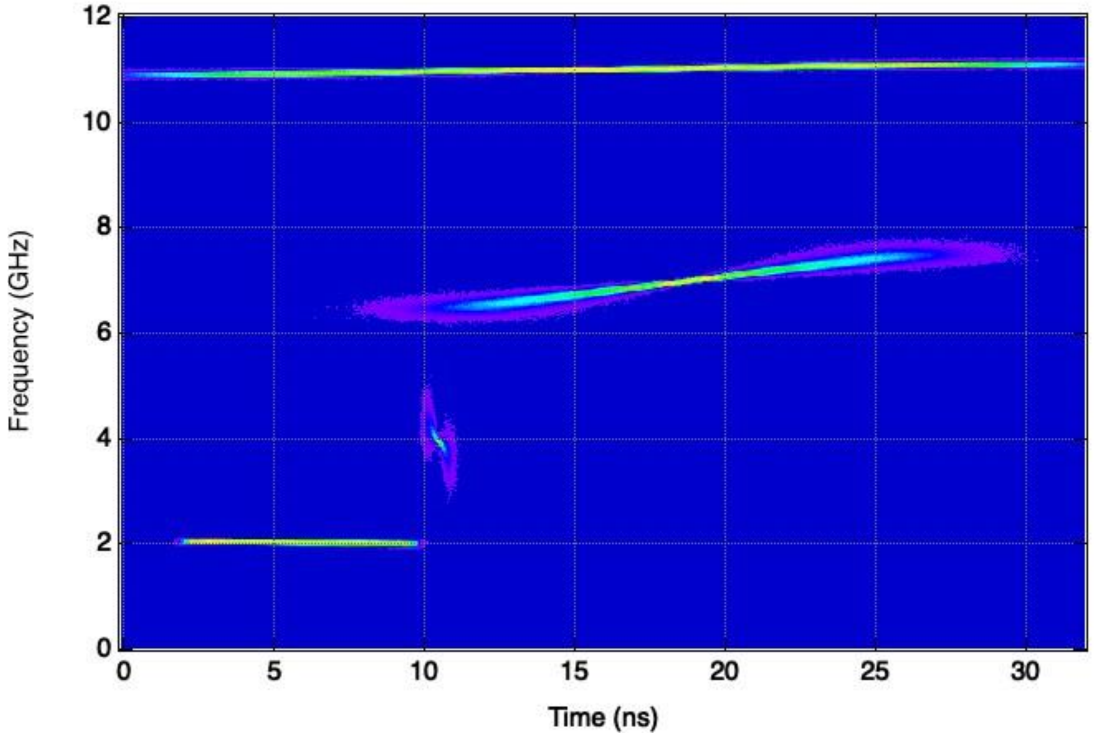
Reduction of Data

- **Enhancement of time resolved features in data**
  - **From a single FFT over a 32ns window we can extract sub-nanosecond resolution of component features.**

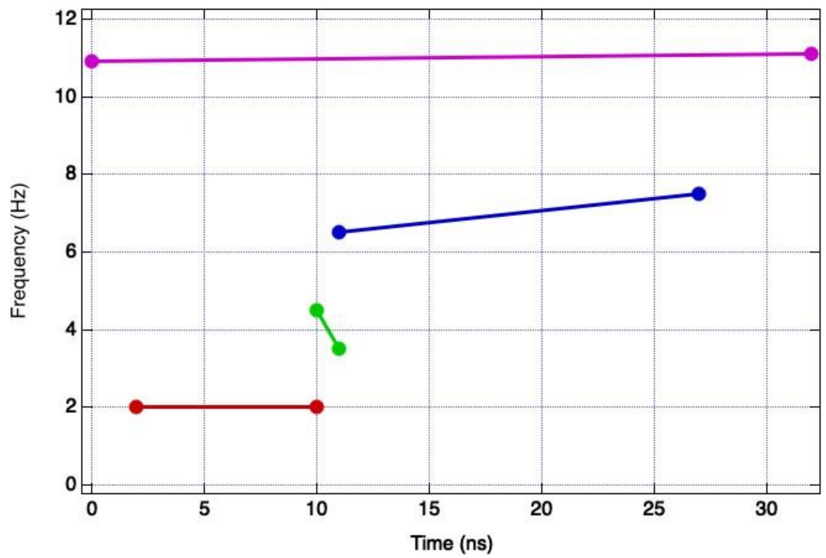
Original – Based on the Fourier transform over the window

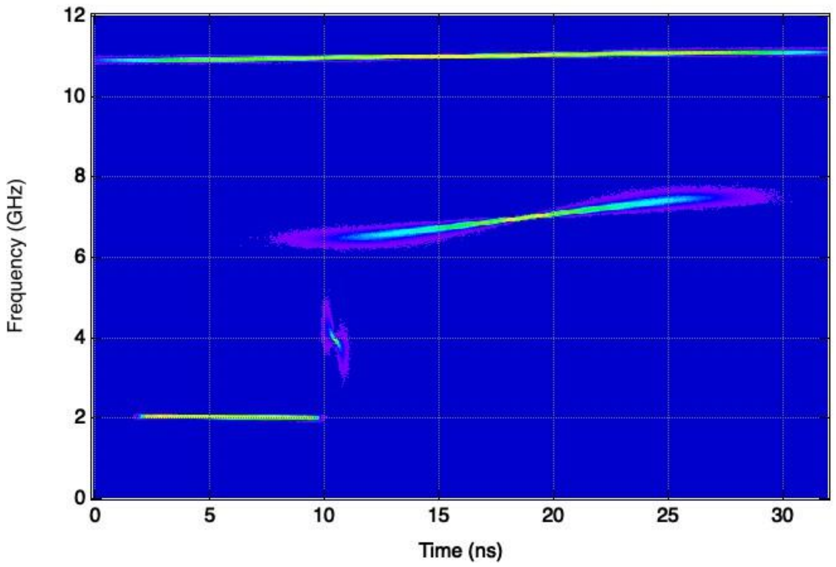


Enhanced – Time dependent features extracted using the phase components of the spectrum



Comparison to Input





Input	Noise	No Noise	Isolated
F0 = 2 T0 = 6 ΔF = 0 ΔT = 4	F0 = 2.000 +/- 0.001 T0 = 5.910 +/- 0.007 ΔF = -0.017 +/- 0.011 ΔT = 3.84 +/- 0.12	F0 = 2.001 +/- 0.001 T0 = 5.928 +/- 0.002 ΔF = 0.0 +/- 0.02 ΔT = 3.79 +/- 0.16	
F0 = 4 T0 = 10.5 ΔF = -0.5 ΔT = 0.5	F0 = 3.940 +/- 0.002 T0 = 10.506 +/- 0.002 ***ΔF = -0.35 +/- 0.24*** ΔT = 0.41 +/- 0.08	F0 = 3.957 +/- 0.002 T0 = 10.503 +/- 0.003 ***ΔF = -0.35 +/- 0.17*** ΔT = 0.43 +/- 0.04	F0 = 3.993 +/- 0.002 T0 = 10.488 +/- 0.001 ***ΔF = -0.54 +/- 0.13*** ΔT = 0.51 +/- 0.06
F0 = 7 T0 = 19 ΔF = 0.5 ΔT = 8	F0 = 6.973 +/- 0.004 T0 = 18.78 +/- 0.03 ΔF = 0.55 +/- 0.08 ΔT = 7.9. +/- 1.0	F0 = 6.991 +/- 0.006 T0 = 19.14 +/- 0.05 ΔF = 0.54 +/- 0.07 ΔT = 8.5. +/- 1.1	
F0 = 11 T0 = 16 ΔF = 0.1 ΔT = 16	F0 = 11.003 +/- 0.002 T0 = 16.28 +/- 0.11 ΔF = 0.11 +/- 0.02 ***ΔT = 14.9 +/- 2.3***	F0 = 10.999 +/- 0.001 T0 = 15.9 +/- 0.1 ΔF = 0.106 +/- 0.015 ***ΔT = 14.9 +/- 1.8***	

Discussion

- **Analysis of the Complex spectrum:**
  - Provides sub-window spectral information
  - Resolves spectral gradients

- **"Not ready for prime time"**
  - Currently requires user interaction
    - Determine peak isolation
    - Limits of χ^2-minimization
  - Where/how does it break?
    - How strict are the requirements in the signal quanta framework?
  - How to implement analysis in a moving window
    - Output is an unconventional format.